Design of LDPC Codes Based on Progressive Edge Growth Techniques for Block Fading Channels

André G. D. Uchôa, Cornelius Healy, Rodrigo C. de Lamare, and Richard D. Souza

***Abstract*—A novel algorithm to design Root-Check LDPC codes based on Progressive Edge Growth (PEG) techniques for block-fading channels is proposed. The performance of the new codes is investigated in terms of the Frame Error Rate (FER) and the Bit Error Rate (BER). Numerical results show that the codes constructed by the proposed algorithm outperform codes constructed by the existing methods by 0.5dB.**

***Index Terms*—LDPC, Root-Check, PEG, outage probability.**

I. Introduction

DUE to multi-path propagation and mobility, wireless systems are characterized by time-varying channels with fluctuating signal strength. In applications subject to delay constraints and slowly-varying channels, only limited independent fading realizations are experienced. In such non-ergodic scenarios, the channel capacity is zero since there is an irreducible probability, termed outage probability [1], that the transmitted data rate is not supported by the channel. The block-fading channel is a simple and useful model that captures the essential characteristics of non-ergodic channels [2]. Codes designed for block-fading channels are expected to achieve the limited channel diversity and to offer good coding gains.

In [3] the authors proposed a family of Low-Density Parity-Check codes (LDPC) called Root-LDPC codes for non-ergodic block-fading channels. The Root-LDPC codes are shown to achieve full diversity on block-fading channels and perform close to the outage limit when decoded using the iterative Sum-Product (SP) decoding algorithm. In the bipartite graph representation of Root-LDPC codes, a subset of connections are deterministically selected to guarantee full diversity for information bits and the remaining connections are generated randomly. In [4] the construction of structured Root-LDPC codes is proposed by means of tiling circulant matrices, i.e., by designing Quasi-Cyclic Low-Density Parity-Check (QC-LDPC) codes. It is also shown that the QC-LDPC codes can perform as good as randomly generated Root-LDPC codes over block-fading channels.

It is known that the girth, i.e., the length of the shortest cycle present in the graph of the code, has a significant effect on code performance. For regular codes, the number of independent messages passed in SP decoding is proportional to the girth of the code [5], which means that a better performance is achieved by codes with larger girths. Consequently, a number of code construction methods aim to produce codes of large girth. In [4] an algorithm to obtain QC-LDPC codes with large girths is presented. Among the algorithms capable of producing LDPC codes with high performance for short to moderate lengths is the Progressive Edge Growth (PEG) algorithm [6]. The codes produced by the PEG algorithm exhibit improved performance compared to random construction methods [6].

The contribution of this letter is to present a PEG-based algorithm to design LDPC codes with Root-Check properties, thus providing Root-LDPC codes with larger girths than those obtained by previously reported methods. A strategy that imposes constraints on a PEG-based algorithm which are required by Root-Check LDPC codes is devised. The codes generated by the proposed algorithm can achieve a significantly better performance in terms of FER and BER than previous works [3], [4]. The new design can save up to 0.5dB in terms of signal to noise ratio (SNR) to achieve the same FER or BER when compared to the known designs.

II. System Model

Consider a block fading channel, where F is the number of independent fading blocks per codeword of length N. Following [4], the t-th received symbol is given by:

$$y_t = h_f x_t + n_t, \tag{1}$$

where $t = \{1, 2, \cdots, N\}$, $f = \{1, 2, \cdots, F\}$, f and t are related by $f = \lceil F \frac{t}{N} \rceil$, where $\lceil \phi \rceil$ returns the smallest integer not smaller than ϕ, h_f is the real Rayleigh fading coefficient of the f-th block, x_t is the transmitted signal, and n_t is additive white Gaussian noise with zero mean and variance $N_0/2$. In this paper, we assume that the transmitted symbols x_t are binary phase shift keying (BPSK) modulated. We assume that the receiver has perfect channel state information, and that the SNR is defined as E_b/N_0, where E_b is the energy per information bit. The information transmission rate is $R = K/N$, where K is the number of information bits per codeword of length N. We consider $R = 1/F$, since then it is possible to design a practical diversity achieving code [4].

The performance of a communication system in a non-ergodic block fading channel can be investigated by means of the outage probability [2], [7]–[9], which is defined as:

$$P_{out} = \mathcal{P}(I < R), \tag{2}$$

where $\mathcal{P}(\phi)$ is the probability of event ϕ. The mutual information I_G, supposing Gaussian channel inputs, is [4]:

$$I_G = \frac{1}{F} \sum_{f=1}^{F} \frac{1}{2} \log_2 \left(1 + 2R \frac{E_b}{N_0} h_f^2 \right), \tag{3}$$

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so that an outage occurs when the average accumulated mutual information among blocks is smaller than the attempted information transmission rate.

The mutual information I_{BPSK}, supposing BPSK inputs, does not have a closed form and it is given by [4]:

$$I_{BPSK} = \frac{1}{F}\sum_{f=1}^{F}\frac{1}{2}\cdot\left(g\left(\sqrt{2R\frac{E_b}{N_0}h_f^2}\right)+g\left(-\sqrt{2R\frac{E_b}{N_0}h_f^2}\right)\right), \tag{4}$$

where

$$g(\tau) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(w-\tau)^2}{2}}\log_2\frac{2}{1+e^{-2w\tau}}\mathrm{d}w. \tag{5}$$

The value of $g(\tau)$ can be computed by using the Gauss-Hermite quadrature method. The FER performance of the LDPC codes designed in this paper is compared to the outage probability, considering both Gaussian and BPSK inputs.

III. Proposed Design Algorithm

First of all, we will introduce some definitions and notations. Then, we will present the pseudo-code of our proposed algorithm. In this work the case of a block-fading channel with $F = 2$ is considered. The LDPC code in systematic form is specified by its sparse parity check matrix $\mathbf{H}$:

$$\mathbf{H} = [\mathbf{I}_M \ \mathbf{P}], \tag{6}$$

where $\mathbf{I}_M$ is the identity matrix of size $M = N - K$, and $\mathbf{P}$ is an M-by-K matrix. The generator matrix for the code is $\mathbf{G} = [\mathbf{P}' \ \mathbf{I}_K]$. Following the notation described in [4], the parity check matrix for a Root-Check LDPC code $C(3,6)$ with rate $R = 1/2$, can be specified as:

$$\mathbf{H}_{RC} = \begin{bmatrix} \mathbf{I} & \mathbf{H}_2 & \mathbf{0} & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{I} & \mathbf{H}_3 & \mathbf{0} \end{bmatrix} \tag{7}$$

where $\mathbf{I}$ is an identity matrix of size $\frac{N}{4}$, $\mathbf{H}_2$ is a square matrix with Hamming weight 2 of size $\frac{N}{4}$ and $\mathbf{H}_3$ is a square matrix with Hamming weight 3 of size $\frac{N}{4}$. The Hamming weights 2 and 3 mean that the sum of each column and each row from each square matrix have only 2 and 3 entries with ones [4].

The variable node degree sequence D_s is defined as the set of column weights of the $\mathbf{H}$ designed, and is prescribed by the variable node degree distribution $\lambda(x)$ as described in [10]. Moreover, D_s is arranged in non-decreasing order. The proposed algorithm, called PEG Root-Check, constructs $\mathbf{H}$ by operating progressively on variable nodes to place the edges required by D_s. The Variable Node (VN) of interest is labeled v_j and the candidate check nodes are individually referred to as c_i. The PEG Root-Check algorithm chooses a check node c_i to connect to the variable node of interest v_j by expanding a constrained sub-graph from v_j up to maximum depth l. The set of check nodes found in this sub-graph is denoted $N_{v_j}^l$ while the set of check nodes of interest, those not currently found in the sub-graph, are denoted $\overline{N_{v_j}^l}$. For the PEG Root-Check algorithm, a check node is chosen at random from the minimum weight check nodes of this set.

A. Pseudo-code for the PEG Root-Check Algorithm

Initialization: A matrix of size $M \times N$ is created with identity matrices of size $\frac{M}{2}$ in the positions shown in (7) and zeros in all other positions. We define the indicator vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ as:

$$\mathbf{z}_1 = [\mathbf{0}_{1\times\frac{M}{2}}, \mathbf{1}_{1\times\frac{M}{2}}], \tag{8}$$

$$\mathbf{z}_2 = [\mathbf{1}_{1\times\frac{M}{2}}, \mathbf{0}_{1\times\frac{M}{2}}], \tag{9}$$

which are modeled on the indicator of the original PEG algorithm [6] for tracking $\overline{N_{v_j}^l}$ as the sub-tree expands. The degree sequence as defined for LDPC codes must be altered to take into account the structure imposed by Root-Check codes, namely the identity matrices of (7). The Root-Check code degree sequence is:

$$D_s = [D_{v_1}-1, D_{v_2}-1, \ldots, D_{v_{\frac{N}{2}}}-1,\ D_{v_{\frac{N}{2}+1}}, \ldots, D_{v_N}]. \tag{10}$$

Imposing $\mathbf{z}_1$ and $\mathbf{z}_2$ for appropriate sets of VNs forces the parity-check matrix to have the form of (7). The pseudo-code for our proposed PEG Root-Check algorithm is detailed in Algorithm 1.

Algorithm 1 PEG Root-Check Algorithm for $F = 2$

for $j = 1 : N$ **do**
  for $k = 0 : D_s(j) - 1$ **do**
    if $(j \le \frac{N}{4}) \parallel (\frac{N}{2} < j \le \frac{3N}{4})$ **then**
      $\mathbf{z}_{PEG} = \mathbf{z}_1$
    else
      $\mathbf{z}_{PEG} = \mathbf{z}_2$
    end if
    if $k == 0 \quad \& \quad j > \frac{N}{2}$ **then**
      Choose candidate at random from minimum weight CNs of the CN set indicated by $\mathbf{z}_{PEG}$.
    else
      Expand the tree from the VN of interest under the current setting. As the tree expands, for any CNs newly added to the tree the corresponding entry of $\mathbf{z}_{PEG}$ is set to 0.
      Expand the tree to depth l s.t. the weight of $\mathbf{z}_{PEG}$ stops decreasing but is greater than 0 **or** the weight of $\mathbf{z}_{PEG} \neq 0$ but the weight of $\mathbf{z}_{PEG}$ at the next level l+1, = 0.
      Place the edge (c_i, V_j) randomly among the minimum weight check nodes of the set indicated by $\mathbf{z}_{PEG}$
    end if
  end for
end for

IV. Simulations

The performance of the proposed PEG-Root-Check LDPC codes when used in a Rayleigh block-fading channel with $F = 2$ independent fading blocks is analyzed. All LDPC codes simulated here are (3,6) regular LDPC codes with rate $R = \frac{1}{2}$. The BPSK outage limit in (4) and the Gaussian outage limit in (3) are drawn in dashed line and solid line, respectively, in each figure for reference. In Fig. 1, it is compared the FER

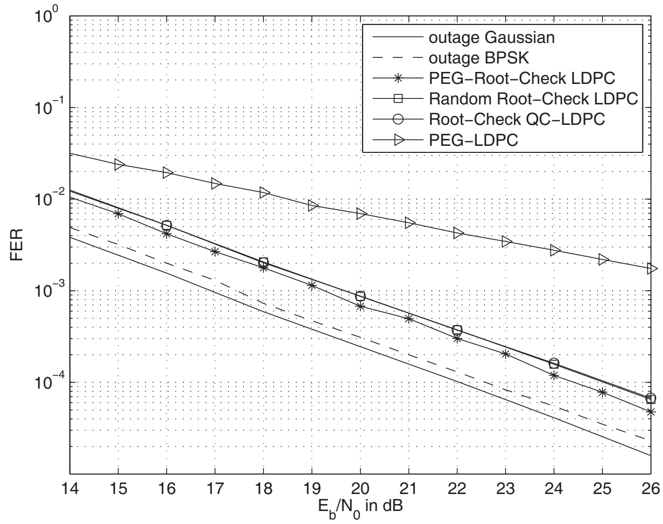

Fig. 1. FER performance comparison for Root-Check QC-LDPC codes, Random Root-Check LDPC codes, PEG LDPC codes and PEG-Root-Check LDPC codes over a block-fading channel with $F = 2$ and $L = 1200$. The maximum number of iterations is 20.

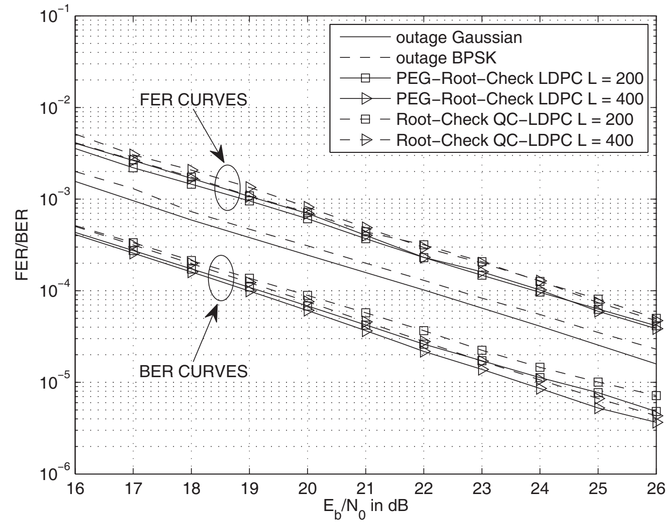

Fig. 2. FER and BER performance comparison for PEG-Root-Check LDPC codes and Root-Check QC-LDPC codes with codeword length $L = 200$ and 400 bits over a block-fading channel with $F = 2$. The maximum number of iterations is 20.

performance among the proposed PEG-Root-Check LDPC codes, Random Root-Check LDPC codes [3] and Root-Check QC-LDPC codes [4]. The codeword length L is 1200 bits. Standard SP algorithm is employed at the decoder with a maximum of 20 iterations. Following [3], [4], a maximum of 20 iterations are enough to obtain a good performance in terms of FER for block-fading channels. The performance of a PEG based LDPC code [6] is also shown. From the results, it can be noted that the proposed PEG-Root-Check LDPC code outperforms the other Root-Check based LDPC codes and it can save up to 0.5dB for the same FER performance. In the case of the PEG-Root-Check LDPC codes the minimum girth is 12. In fact, all Root-Check-based codes are able to achieve the full diversity order of the channel, while PEG LDPC codes fail to achieve full diversity. The curves presented for the proposed PEG-Root-Check LDPC codes show that a Root-Check LDPC code generated with the proposed PEG-based algorithm produces a better performance in terms of FER.

Fig. 2 shows the FER and BER performance of PEG-Root-Check LDPC codes and Root-Check QC-LDPC codes with different codeword lengths. Two codeword lengths $L = 200, 400$, bits are considered. At the high SNR region we can observe that the proposed PEG-Root-Check LDPC codes outperform the Root-Check QC-LDPC codes for the same FER/BER performance. It must be mentioned that for different codeword length we can still save up to 0.5dB in terms of SNR for the same FER/BER performance. Moreover, note that the proposed design is less than 1.5 dB away from the theoretical limit given by the BPSK outage probability.

V. Conclusion

A novel PEG-based algorithm has been proposed to design Root-Check LDPC codes. Based on simulations, the proposed method was compared to Random Root-Check LDPC and Root-Check QC-LDPC codes. The results demonstrate that the PEG-Root-Check LDPC codes generated by our proposed algorithm outperform the previously reported codes [3], [4] in a wide range of SNR values and provide a gain of up to 0.5 dB. Moreover, it must be mentioned that a PEG-based code can be structured for linear complexity [11] so that the gains come at no extra cost in the operation of the code.

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