# Set-Membership Adaptive Constant Modulus Algorithm with Generalized Sidelobe Canceler Based on Dynamic Bounds for Beamforming☆

Yunlong Cai[a,*], Rodrigo C. de Lamare[b], Minjian Zhao[a]

[a] *Department of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310027, China*
[b] *CETUC, PUC-Rio, Brazil and Communications Research Group, Department of Electronics, University of York, YO10 5DD York, U.K.*

**Abstract**

In this work, we propose an adaptive set-membership constant modulus (SM-CM) algorithm with a generalized sidelobe canceler (GSC) structure for blind beamforming. We develop a stochastic gradient (SG) type algorithm based on the concept of SM filtering for adaptive implementation. The filter weights are updated only if the constraint cannot be satisfied. In addition, we also propose an extension of two schemes of time-varying bounds for beamforming with a GSC structure and incorporate parameter and interference dependence to characterize the environment which improves the tracking performance of the proposed algorithm in dynamic scenarios. A convergence analysis of the proposed adaptive SM filtering techniques is carried out. Simulation results show that the proposed adaptive SM-CM-GSC algorithm with dynamic bounds achieves superior performance to previously reported methods at a reduced update rate.

*Keywords:* Adaptive beamforming, set-membership filtering, constant modulus algorithm, interference suppression, generalized sidelobe canceler.

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*Corresponding author

*Email addresses:* `ylcai@zju.edu.cn` (Yunlong Cai), `rcdl500@ohm.york.ac.uk` (Rodrigo C. de Lamare), `mjzhao@zju.edu.cn` (Minjian Zhao)

## 1. Introduction

Blind beamforming has been widely applied to system identification, localization and interference suppression in communications and array processing systems [1]-[8]. It is often employed with receivers equipped with an antenna array to steer a directional beampattern towards the desired user and suppress interference without the need for training sequence or pilots in spatial filtering. In these situations, beam-width and sidelobe levels are the important characteristics of the response and give rise to various beamformer structures, i.e., multiple sidelobe canceller (MSC) and the generalized sidelobe canceller (GSC) [9]-[11]. In particular, the beamformer with a GSC structure that employs a main branch along with a group of auxiliary branches[1]has attracted significant attention.

An important issue is the choice of a suitable criterion for the design of the beamformer. The constrained minimum variance (MV) and the constrained constant modulus (CM) criteria are considered as the most promising design approaches due to their simplicity and effectiveness. The MV-based algorithms are designed in such a way that they attempt to minimize the filter output power while maintaining a constant response in the direction of a signal of interest [2]-[4], [12], [13]. In fact, the CM-based algorithms are based on a criterion that penalizes deviations of the modulus of the received signal away from a fixed value and forced to satisfy one or a set of linear constraints such that signals from the desired user are detected [8], [14]-[17]. The literature indicates that the CM-based algorithms outperform the MV-based algorithms and lead to a solution comparable to that obtained from the minimization of the mean squared error (MSE). Furthermore, the CM-type algorithms are robust against estimation errors and prevent a severe performance degradation in the presence of uncertainties [14]-[17]. Moreover, it is worth pointing out that the beamforming algorithms that incorporate the GSC structures using the CM and MV criteria were proposed in [8] and [3], respectively. The results showed that the GSC-based blind beamforming algorithms lead to an improved performance

[1]In this respect, the interference is assumed to be presented in both main and auxiliary branches, while the desired signal is available in the main branch due to its high gain in the direction of interest [1]. The auxiliary branches are used to form an estimate of the main branch interference, which is subtracted from the output of the main branch in order to generate the final estimate of the desired signal.

compared to the algorithms with a direct-form processor (DFP).

In practice, the beamformer weights must be continually adapted over time in order to cope with changes in the radio signal environment [12], [28]. Therefore, it is preferable to implement blind beamformers with adaptive filtering algorithms such as stochastic gradient (SG) algorithms [1]. For this reason the improvement of blind adaptive SG techniques is an important research and development topic. One problem for the adaptive SG algorithms is that their performance is strongly dependent on the choice of the step-size value [1]. Another problem is the computational complexity associated with the adaptation for every time instant. Set-membership (SM) filtering techniques have been proposed to address these issues [29]-[40]. They specify a bound on the magnitude of the estimation error or the array output, and can reduce the complexity due to data-selective updates. From [29]-[40], we can see that the SM filtering techniques are able to achieve a reduction in computation without performance degradation compared to conventional algorithms due to the use of an adaptive step-size for each update. In particular, the work in [31] appears to be the first approach to combine the SM filtering algorithm with the CM criterion. Furthermore, in nonstationary wireless environments, interferers frequently enter and exit the system, making it very difficult for the SM filtering algorithms to compute a predetermined error bound and the risk of overbounding and underbounding is significantly increased. Hence, the performance of SM filtering algorithms strongly depends on the error bound specification, which motivates several SM algorithms with time-varying bound schemes [34]-[37].

In this work, we present extensions of the methods reported in [34] to the GSC structure using blind adaptive set-membership constrained constant modulus (SM-CM) algorithms for beamforming. Simulation results show that the proposed adaptive SM-CM beamforming algorithm realized in the GSC structure (SM-CM-GSC) with dynamic bounds achieves superior performance to previously reported methods at a reduced update rate. Compared to the existing SM algorithms the contributions of this work are summarized as follows:

1. To the best of our knowledge, there is a very small number of adaptive blind beamforming algorithms with SM techniques. We develop an SG-type adaptive CM beamforming algorithm based on the concept of SM filtering that exploits the GSC structure.
2. The filter weights of the proposed algorithm are updated only if the con-

straint cannot be satisfied. Therefore, it significantly reduces the computational complexity due to the sparse updates compared to the conventional adaptive CM-GSC beamforming algorithms.

3. The bounding schemes of the existing SM filtering algorithms cannot be applied to the proposed adaptive blind beamforming algorithm in nonstationary scenarios. We propose two schemes of time-varying bounds for beamforming with a GSC structure and incorporate parameter and interference dependence to characterize the environment for improving the tracking performance in dynamic scenarios.
4. A convergence analysis of the proposed adaptive SM filtering techniques is carried out and analytical expressions to predict the steady-state MSE are obtained.

In this paper, the superscripts $(.)^T$, $(.)^*$, $(.)^{-1}$, and $(.)^H$ denote transpose, element-wise conjugate, matrix inverse, and Hermitian transpose, respectively. Bold symbols denote matrices or vectors. The symbols $E[.]$, $|.|$, $||.||$, $\mathbf{I}$ and $\mathbf{0}$ represent the expectation operator, the norm of a scalar, the norm of a vector, an identity matrix of appropriate dimension and a zero vector of appropriate dimension, respectively.

The remainder of this paper is organized as follows: we briefly describe a system model for beamforming and the design of CM beamformers with a GSC structure in Section 2. The SM filtering framework and the adaptive blind SM-CM-GSC algorithm are introduced in Section 3. Section 4 introduces two strategies to compute time-varying bounds for the proposed algorithms. Convergence analysis of the resulting algorithm and the analytical formulas to predict the steady-state MSE are developed in Section 5. The simulation results are presented in Section 6. Finally, Section 7 draws the conclusions.

## 2. System Model and Linearly CM-GSC Beamformer

Let us suppose that $q$ narrowband signals impinge on a uniform linear array (ULA) of $m$ $(m \geq q)$ sensor elements. The sources are assumed to be in the far field with direction of arrivals (DOAs) $\theta_0, \ldots, \theta_{q-1}$. The $i$th snapshot's received vector $\mathbf{r} \in \mathcal{C}^{m\times 1}$ can be modeled as

$$\mathbf{r}(i) = \mathbf{A}(\boldsymbol{\theta})\mathbf{b}(i) + \mathbf{n}(i), \tag{1}$$

where $\boldsymbol{\theta} = [\theta_0, \ldots, \theta_{q-1}]^T \in \mathcal{R}^{q\times 1}$ is the vector with the DOAs of the signals, $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_0), \ldots, \mathbf{a}(\theta_{q-1})] \in \mathcal{C}^{m\times q}$ comprises the normalized signal steering vectors $\mathbf{a}(\theta_k) \in \mathcal{C}^{m\times 1}$

$$\mathbf{a}(\theta_k) = \frac{1}{\sqrt{m}}[1, e^{-2\pi j \frac{u}{\lambda_c}\cos(\theta_k)}, \ldots, e^{-2\pi j(m-1)\frac{u}{\lambda_c}\cos(\theta_k)}]^T, \tag{2}$$

where $k = 0, \ldots, q-1$, $\lambda_c$ is the wavelength, $u$ ($u = \frac{\lambda_c}{2}$ in general) is the inter-element distance of the ULA. To avoid mathematical ambiguities, the steering vectors $\mathbf{a}(\theta_k)$ are assumed to be linearly independent, $\mathbf{b}(i) = [b_0(i), b_1(i), \ldots, b_{q-1}(i)]^T$ is the source data vector, where we assume that the signals are independent and identically distributed (i.i.d) random variables with equal probability from the set $\{\pm 1\}$. The vector $\mathbf{n} \in \mathcal{C}^{m\times 1}$ is a Gaussian noise with $E[\mathbf{n}\mathbf{n}^H] = \sigma_n^2\mathbf{I}$, where $\sigma_n^2$ denotes the noise variance. In this work, we assume that $\theta_0$ corresponds to the direction of the desired user with respect to the antenna arrays and is known beforehand by the beamformer. In practice, $\theta_0$ can be estimated by DOA estimation algorithms. The output of a narrowband GSC beamformer is given by

$$y(i) = \tilde{\mathbf{w}}^H(i)\mathbf{r}(i), \tag{3}$$

where $\tilde{\mathbf{w}}(i) = v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}(i)$, $\mathbf{B} \in \mathcal{C}^{(m-1)\times m}$ denotes the signal blocking matrix[2], $\mathbf{w}(i) = [w_1, \ldots, w_{m-1}]^T \in \mathcal{C}^{(m-1)\times 1}$ is the complex weight vector of the filter and $v$ is a real constant. That is to say, the GSC structure consists of a main branch and an auxiliary branch. The output of the main branch is $v\mathbf{a}^H(\theta_0)\mathbf{r}(i)$, and the output of the auxiliary branch is $\big(\mathbf{B}^H\mathbf{w}(i)\big)^H\mathbf{r}(i)$. The auxiliary branch is employed to form an estimate of the main branch interference that can be used for cancelation.

The CM-GSC optimization problem determines the filter parameters $\mathbf{w}(i)$ by solving

$$\text{minimize} \quad J_{CM}(\mathbf{w}(i)) = E\Big[\big(|\tilde{\mathbf{w}}^H(i)\mathbf{r}(i)|^2 - 1\big)^2\Big]. \tag{4}$$

The objective of (4) is to minimize the expected deviation of the squared modulus of the beamformer output to a constant while maintaining the contribution from $\theta_0$ constant, i.e. $\tilde{\mathbf{w}}^H(i)\mathbf{a}(\theta_0) = (v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}(i))^H\mathbf{a}(\theta_0) = v$. The CM-

[2] It is obtained by the singular value decomposition (SVD) or the QR decomposition algorithms and collecting eigenvectors corresponding to null eigenvalues [41]. Thus, $\mathbf{B}\mathbf{a}(\theta_0) = \mathbf{0}$ means that the term $\mathbf{B}$ effectively blocks any signal coming from the look direction $\theta_0$.

GSC design can have its convexity enforced by adjusting the parameter $v$, note that the detailed analysis of the optimization problem is shown in the Appendix. The CM-GSC filter expression that iteratively solves the problem in (4) is given by

$$\mathbf{w}(i+1) = \left(E[|y(i)|^2\mathbf{B}\mathbf{r}(i)\mathbf{r}^H(i)\mathbf{B}^H]\right)^{-1} E[(v\mathbf{a}^H(\theta_0)y^*(i)\mathbf{r}(i) - 1)^* y^*(i)\mathbf{B}\mathbf{r}(i)], \quad (5)$$

where $y(i) = \tilde{\mathbf{w}}^H(i)\mathbf{r}(i) = v\mathbf{a}^H(\theta_0)\mathbf{r}(i) - \mathbf{w}^H(i)\mathbf{B}\mathbf{r}(i)$. It should be remarked that the expression in (5) is a function of previous values of filter $\mathbf{w}(i)$ and therefore must be iterated in order to reach a solution. However, the method of computing (5) is not practical in wireless communications applications with mobile users and nonstationary interferers, and hence an adaptive implementation is needed. The SG algorithm is one of the most widely used adaptive algorithms, but one problem with this algorithm is the computational complexity related to the adaptation for each snapshot. In order to reduce the update rate and improve the convergence performance, we will introduce the proposed adaptive low-complexity SM beamforming algorithm in the following section.

## 3. Proposed Adaptive SM Technique

In this work, we develop the SM-CM-GSC adaptive SG algorithm that updates the filter weights only if the bound constraint $e^2(i) \leq \gamma^2$ cannot be satisfied, where $e(i) = |\tilde{\mathbf{w}}^H(i)\mathbf{r}(i)|^2 - 1$ denotes the prediction error and $\gamma$ denotes a specified bound. The solution of the proposed algorithm is a set in the parameter space [38], which includes some estimates that satisfy the bound constraint corresponding to different $\mathbf{r}$ for different time instants.

### *3.1. Proposed SM Framework*

Let us define a sample space $\mathbb{S}$ that contains all possible data $\{\mathbf{r}\}$. Then, we define the *feasibility set* $\mathcal{Q}$ as

$$\mathcal{Q} = \bigcap_{\mathbf{r}\in\mathbb{S}} \{\mathbf{w} \in \mathcal{C}^{(m-1)\times 1} : (|\tilde{\mathbf{w}}^H\mathbf{r}|^2 - 1)^2 \leq \gamma^2\}, \quad (6)$$

which contains the values that fulfill the error bound.

We apply the feasibility set to a time-varying scenario; therefore, it contains all estimates that fulfill the bound constraint at the $i$th time instant. This set is termed the *constraint set* and is given by

$$\mathcal{H}_i = \{\mathbf{w} \in \mathcal{C}^{(m-1)\times 1} : (|\tilde{\mathbf{w}}^H \mathbf{r}(i)|^2 - 1)^2 \leq \gamma^2\}. \tag{7}$$

Our aim is to develop an adaptive algorithm that updates the parameters such that it will always remain within the constraint set.

As depicted in Fig. 1, the proposed adaptive scheme introduces the principle of the SM filtering technique into the blind CM beamforming algorithm with a GSC structure. Thus, it operates with respect to certain snapshots and therefore has a reduced computational complexity. Furthermore, the data-selective updates will lead to highly effective variable step-size for the SG-based SM beamforming algorithm. In the following, we will describe the proposed blind adaptive algorithm in detail.

### *3.2. Proposed SM-CM-GSC Adaptive Algorithm*

We devise a gradient descent strategy to compute the filter weight vector $\mathbf{w}$ that minimizes the instantaneous CM-GSC cost function, the adaptation is required when the square of the error $e^2(i)$ exceeds a specified error bound $\gamma^2(i)$. Note that the bound here can be assumed to be time-varying and based on the estimated parameters of the filter weight vector, and the time-varying bound schemes will be addressed in the next section. The problem is formulated as follows,

$$\text{minimize} \qquad J_{CM} = \big(|\tilde{\mathbf{w}}^H(i)\mathbf{r}(i)|^2 - 1\big)^2, \tag{8}$$

$$\text{whenever} \quad e^2(i) > \gamma^2(i). \tag{9}$$

We consider the following gradient search procedure:

$$\mathbf{w}(i+1) = \mathbf{w}(i) - \mu(i)\frac{\partial J_{CM}}{\partial \mathbf{w}^*}, \tag{10}$$

where $\mu(i)$ is the effective variable step-size. By taking the gradient of (8) with respect to $\mathbf{w}^*$ we have $\frac{\partial J_{CM}}{\partial \mathbf{w}^*} = -|y(i)|^2\mathbf{B}\mathbf{r}(i)y^*(i) + \mathbf{B}\mathbf{r}(i)y^*(i)$. Then, we obtain the following SG algorithm

$$\mathbf{w}(i+1) = \mathbf{w}(i) - \mu(i)\Big(\mathbf{B}\mathbf{r}(i)y^*(i) - |y(i)|^2\mathbf{B}\mathbf{r}(i)y^*(i)\Big). \tag{11}$$

The variable step-size value will attempt to find the shortest path from $\mathbf{w}(i)$ to the bounding hyperplane of $\mathcal{H}_i$ in accordance with the principle of minimal disturbance. In other words, $\mathbf{w}(i+1)$ is the projection of $\mathbf{w}(i)$ on $\mathcal{H}_i$. However, if $\mathbf{w}(i) \in \mathcal{H}_i$, we can see that the error bound constraint is satisfied; therefore, no update is necessary, and $\mathbf{w}(i+1) = \mathbf{w}(i)$. Note that the constraint set comprises of two parallel hyper-strips in the parameter space. Based on the constraint $e^2(i) > \gamma^2(i)$, we consider the following two cases for update: 1) $|y(i)| > \sqrt{1+\gamma(i)}$ and 2) $|y(i)| < \sqrt{1-\gamma(i)}$, and obtain the following expression for $\mu(i)$:

$$\mu(i) = \begin{cases} \left(1 - \frac{\sqrt{1+\gamma(i)}}{|y(i)|}\right) \frac{1}{(\mathbf{r}^H(i)\mathbf{B}^H|y(i)|^2 - \mathbf{r}^H(i)\mathbf{B}^H)\mathbf{B}\mathbf{r}(i)} & \text{if} \quad |y(i)| > \sqrt{1+\gamma(i)} \\ \left(1 - \frac{\sqrt{1-\gamma(i)}}{|y(i)|}\right) \frac{1}{(\mathbf{r}^H(i)\mathbf{B}^H|y(i)|^2 - \mathbf{r}^H(i)\mathbf{B}^H)\mathbf{B}\mathbf{r}(i)} & \text{if} \quad |y(i)| < \sqrt{1-\gamma(i)} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where the derivations are detailed in Appendix C. The proposed SM-CM-GSC adaptive algorithm which consists of equations (11) and (12) updates the filter vector $\mathbf{w}(i)$ over time in a manner to converge to the optimum filter weight vector corresponding to (5). The SM filtering technique with a time-varying bound is employed to determine a set of estimates $\{\mathbf{w}(i)\}$ that satisfy the bounded constraint.

Note that the blocking matrix $\mathbf{B}$ has no particular structure if the SVD or QR decomposition is employed. Therefore, the complexity of computing the error $y(i)$ is high, since computing the error involves carrying out the multiplication $\mathbf{B}\mathbf{r}(i)$. However, there are several ways to easily bypass this computational problem [42]-[45]. One method is the application of the correlation subtractive structure (CSS) [42], [44]. The commonly used blocking matrix with the CSS implementation is given by $\mathbf{B} = \mathbf{I} - \mathbf{a}(\theta_0)\mathbf{a}^H(\theta_0)$. By using the particular structure of the blocking matrix, we can compute the error with a linear complexity. For each snapshot, the conventional adaptive CM-GSC beamforming algorithm requires $3m$ multiplications and $3m-1$ additions, while the proposed SM-CM-GSC adaptive beamforming algorithm requires $2m + \eta m$ multiplications and $2(m-1) + \eta(m+1)$ additions, where $0 < \eta \leq 1$ denotes the update rate. In particular, for a configuration with $m = 40$ and $\eta = 20\%$, the number of multiplications for the conventional CM-GSC and the proposed SM-based algorithms are 120 and 88, respectively. The number of additions for them are 119 and 86, respectively. It is worth mentioning that the computational complexity is reduced significantly due to the data-selective updates.

## 4. Time-Varying Bound Schemes

The bound of SM filtering algorithms is an important quantity to measure the quality of the estimates that could be included in the constraint set. In [36], [37], several predetermined bounding schemes have been reported for development of the adaptive SM filters, which achieve reduced complexity without performance degradation. However, in a nonstationary scenario, they are impractical to reflect the time-varying nature of the environment and may result in poor convergence and tracking performance. To the best of our knowledge, there is a very small number of works employing time-varying bounds for SM filters. In this work, we present extensions of the methods reported in [34] to the GSC scheme to compute the time-varying error bound $\gamma(i)$, which is a single coefficient to check if the filter update is carried out or not.

### *4.1. Parameter Dependent Bound (PDB)*

The proposed time-varying bound schemes can increase the convergence and tracking performance. The first scheme is called parameter dependent bound (PDB). It computes a bound for the SM-CM-GSC adaptive algorithm and is given by

$$\gamma(i+1) = (1-\rho)\gamma(i) + \rho\sqrt{\lambda||\tilde{\mathbf{w}}(i)||^2\hat{\sigma}_n^2(i)}, \tag{13}$$

where $\rho$ is a forgetting factor parameter that should be set to guarantee a proper time-averaged estimate of the evolution of the power of GSC beamforming vector $\tilde{\mathbf{w}}(i) = v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}(i)$, $\lambda$ ($\lambda > 1$) is a tuning coefficient and $\hat{\sigma}_n^2(i)$ is an estimate of the noise power. We assume that the noise power is known beforehand at the receiver. The time-varying bound provides a smoother evolution of the weight vector trajectory and thus avoids too high or low values of the squared norm of the weight vector. It establishes a relation between the estimated parameters and the environmental coefficients.

### *4.2. Parameter and Interference Dependent Bound (PIDB)*

The second time-varying bound scheme has a slightly increased complexity compared to the PDB scheme. It combines the PDB with the interference estimation that is provided by the auxiliary branch of the GSC structure. It provides more information about the environment for parameter estimation and has an improved performance. We refer to it as parameter and interference dependent bound (PIDB), and it is an extension of [34] for beamforming design with the CM-GSC criterion. The proposed SM-CM-GSC adaptive beamforming algorithm with the PIDB structure is shown in Fig. 2.

Since the matrix $\mathbf{B}$ blocks the signal which comes from the desired direction, the auxiliary branch of the GSC structure generates the estimate of the interference and the noise. The power of the interference and the noise is given by

$$E[|\mathbf{w}^H(i)\mathbf{B}\mathbf{r}(i)|^2] = \mathbf{w}^H(i)\mathbf{B}\Big(\sum_{k=1}^{q-1}\mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k) + \sigma_n^2\mathbf{I}\Big)\mathbf{B}^H\mathbf{w}(i). \quad (14)$$

By using time averages of the instantaneous values, we can obtain an estimate of (14), which is

$$\nu(i+1) = (1-\rho)\nu(i) + \rho|\mathbf{w}^H(i)\mathbf{B}\mathbf{r}(i)|^2, \quad (15)$$

the component $\nu(i)$ performs the estimate of the interference and the noise power, $\rho$ is a forgetting factor to ensure a proper time-averaged estimate. By incorporating the information of the interference and noise power into the bounding scheme we have the PIDB expression

$$\gamma(i+1) = (1-\rho)\gamma(i) + \rho\big(\sqrt{\psi\nu(i)} + \sqrt{\lambda||\tilde{\mathbf{w}}(i)||^2\hat{\sigma}_n^2(i)}\big), \quad (16)$$

where $\psi$ is a weighting parameter that should be set, note that update equation (15) avoids instantaneous values that are undesirably too high or too low, and thus avoids inappropriate estimates of $\gamma(i)$. Compared with (13), the PIDB involves the estimate of the interference and the noise power and provides more information to track the characteristics of the environments.

## 5. Analysis of the Proposed Algorithm

In this section, we investigate the convergence behavior of our proposed SM schemes when used in the adaptive CM-GSC beamforming algorithm in terms of the steady-state excess MSE. The nonlinearities in the update equations of the CM-GSC beamformer usually lead to significant difficulties in the study of their performance. We use a very efficient approach named energy conservation principle [46]-[49] which overcomes many of these difficulties.

### 5.1. The Range of Step-Size Values for Convergence

In this part, we discuss the range of the step-size values for convergence. In order to do the analysis, we need to write the proposed beamforming filter weights update equation. Let us recall (11), by multiplying $-\mathbf{B}^H$ and adding $v\mathbf{a}(\theta_0)$ on both sides we have

$$\begin{aligned}\tilde{\mathbf{w}}(i+1) &= \tilde{\mathbf{w}}(i) - \mu(i)e(i)\mathbf{r}^H(i)\tilde{\mathbf{w}}(i)\mathbf{B}^H\mathbf{B}\mathbf{r}(i) \\ &= (\mathbf{I} - \mu(i)e(i)\mathbf{d}(i)\mathbf{r}^H(i))\tilde{\mathbf{w}}(i).\end{aligned} \tag{17}$$

Further, we obtain

$$\begin{aligned}\boldsymbol{\varepsilon}(i+1) &= \tilde{\mathbf{w}}_{opt} - \tilde{\mathbf{w}}(i+1) \\ &= (\mathbf{I} - \mu(i)e(i)\mathbf{d}(i)\mathbf{r}^H(i))\boldsymbol{\varepsilon}(i) + \mu(i)e(i)\mathbf{d}(i)\mathbf{r}^H(i)\tilde{\mathbf{w}}_{opt},\end{aligned} \tag{18}$$

where $\mathbf{d}(i) = \mathbf{B}^H\mathbf{B}\mathbf{r}(i)$, $\tilde{\mathbf{w}}_{opt} = v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}_{opt}$ denotes the optimum beamformer, and $\mathbf{w}_{opt}$ denotes the optimum filter for $\mathbf{w}$. By taking expectations on both sides of (18) we have

$$E[\boldsymbol{\varepsilon}(i+1)] = (\mathbf{I} - E[\mu(i)]\mathbf{R}_{dr}(i))E[\boldsymbol{\varepsilon}(i)], \tag{19}$$

where $\mathbf{R}_{dr}(i) = E[e(i)\mathbf{d}(i)\mathbf{r}^H(i)]$ and $\mathbf{R}_{dr}(i)\tilde{\mathbf{w}}_{opt} \approx \mathbf{0}$ [12]. Therefore, it can be concluded that $\tilde{\mathbf{w}}$ converges to $\tilde{\mathbf{w}}_{opt}$ and (19) is stable if and only if $\Pi_{i=0}^{\infty}(\mathbf{I} - E[\mu(i)]\mathbf{R}_{dr}) \rightarrow 0$, which is a necessary and sufficient condition for $\lim_{i\rightarrow\infty} E[\boldsymbol{\varepsilon}(i)] = \mathbf{0}$ and $E[\tilde{\mathbf{w}}(i)] \rightarrow \tilde{\mathbf{w}}_{opt}$. For stability, a sufficient condition for (19) to hold implies that [1]

$$0 \leq E[\mu(\infty)] < \min_k \frac{2}{|\lambda_k^{dr}|}, \tag{20}$$

where $\lambda_k^{dr}$ is the $k$th eigenvalue of $\mathbf{R}_{dr}$ that is not real since it is not symmetric.

*5.2. Steady-State Analysis for Excess MSE*

Let us define the MSE at time index $i$ using the following expression

$$\begin{aligned}\xi(i) &= E\big[|b_0(i) - \tilde{\mathbf{w}}^H(i)\mathbf{r}(i)|^2\big]\\ &= E\big[|b_0(i) - (\tilde{\mathbf{w}}_{opt} - \boldsymbol{\varepsilon}(i))^H\mathbf{r}(i)|^2\big]\\ &= \xi_{min} + E[|e_a(i)|^2] + \mathbf{a}^H(\theta_0)E[\boldsymbol{\varepsilon}(i)] + E[\boldsymbol{\varepsilon}^H(i)]\mathbf{a}(\theta_0)\\ &\quad - E[\tilde{\mathbf{w}}_{opt}^H\mathbf{r}(i)\mathbf{r}^H(i)\boldsymbol{\varepsilon}(i)] - E[\boldsymbol{\varepsilon}^H(i)\mathbf{r}(i)\mathbf{r}^H(i)\tilde{\mathbf{w}}_{opt}]\end{aligned} \tag{21}$$

where we have $\xi_{min} = E[|b_0(i) - \tilde{\mathbf{w}}_{opt}^H\mathbf{r}(i)|^2]$ and $e_a(i) = \boldsymbol{\varepsilon}^H(i)\mathbf{r}(i)$ which denotes the error in the beamformer coefficients $\tilde{\mathbf{w}}(i)$ via the *a priori* estimation error.

When $i$ becomes a large number, since $\tilde{\mathbf{w}}(i) \rightarrow \tilde{\mathbf{w}}_{opt}$ and $E[\boldsymbol{\varepsilon}(i)] \rightarrow 0$ we have the steady-state MSE

$$\lim_{i\rightarrow\infty} \xi(i) = \xi_{min} + \lim_{i\rightarrow\infty} E[|e_a(i)|^2]. \tag{22}$$

Then, we define the steady-state excess MSE:

$$\xi_{ex} = \lim_{i\rightarrow\infty} E[|e_a(i)|^2]. \tag{23}$$

In the following, we derive the expression for $\xi_{ex}$. Based on the energy conservation principle [46]-[49], in the steady state we have the energy preserving equation which is given as follows

$$E\big[\bar{\mu}(i)|e_a(i)|^2\big] = E\Big[\bar{\mu}(i)\Big|e_a(i) - \frac{\mu(i)}{\bar{\mu}(i)}F_e^*(i)\Big|^2\Big], \tag{24}$$

where $\bar{\mu}(i) = 1/||\mathbf{Br}(i)||^2$, $F_e^*(i) = \frac{e_a(i)-e_p(i)}{\mu(i)||\mathbf{Br}(i)||^2}$, $e_p(i) = \boldsymbol{\varepsilon}^H(i+1)\mathbf{r}(i)$ and $y(i) = \tilde{\mathbf{w}}^H(i)\mathbf{r}(i) = (\tilde{\mathbf{w}}_{opt} - \boldsymbol{\varepsilon}(i))^H\mathbf{r}(i) = \tilde{\mathbf{w}}_{opt}^H\mathbf{r}(i) - e_a(i) = b_0(i) + \bar{I}(i) + \bar{n}(i) - e_a(i)$, where $\bar{I}(i)$ and $\bar{n}(i)$ denote the residual interference and the residual noise, respectively, as the output components of the optimum beamformer.

By expanding the right hand side (RHS) of (24), we have

$$\begin{aligned}&E[\mu(i)]E[e_a^*(i)y(i)(1-|y(i)|^2)]\\ &\qquad + E[\mu(i)]E[e_a(i)y^*(i)(1-|y(i)|^2)]\\ &\qquad = E[\mu^2(i)]E[||\mathbf{Br}(i)||^2\underbrace{|y(i)|^2(1-|y(i)|^2)^2}_{|F_e(i)|^2}].\end{aligned} \tag{25}$$

Based on the analytical works in [49] and [16], we also make the following assumptions:

1. In the steady state, the quantities $\{b_0(i), \bar{I}(i), \bar{n}(i), e_a(i)\}$ are zero-mean random variables, and they are mutually independent. The residual interference and the residual noise are Gaussian random variables.
2. In the steady state, $||\mathbf{Br}(i)||^2$ and $|F_e(i)|^2$ are uncorrelated.
3. We have $E[b_k^{2l}(i)] = 1$ for any positive integer $l$.

Further, by employing the assumptions and substituting $y(i) = b_0(i) + \bar{I}(i) + \bar{n}(i) - e_a(i)$ into (25) we have

$$\begin{aligned} & E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]K_1 E[|e_a(i)|^2] \\ & \quad + 3E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]\sigma_I^2 E[|e_a(i)|^4] \\ & \quad + 3E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]\sigma_n^2 E[|e_a(i)|^4] \\ & \quad + E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]E[|e_a(i)|^4] \\ & \quad + E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]K_2 \\ & \quad + E[\mu^2(i)]E[||\mathbf{Br}(i)||^2]E[|e_a(i)|^6] \\ & = 2E[\mu(i)]\big(\sigma_I^2 E[|e_a(i)|^2] + \sigma_v^2 E[|e_a(i)|^2] + E[|e_a(i)|^4]\big), \end{aligned} \tag{26}$$

where $K_1 = 3 + 3\sigma_I^4 + 6\sigma_I^2\sigma_v^2 + 3\sigma_v^4$, $K_2 = \sigma_v^6 + 3\sigma_I^2\sigma_v^4 + 3\sigma_I^4\sigma_v^2 + \sigma_I^6 + \sigma_v^4 + 2\sigma_I^2\sigma_v^2 + \sigma_I^4 + 4\sigma_v^2 + 2\sigma_I^2 + 2$, $\sigma_I^2 = E[\bar{I}^2(i)]$, and $\sigma_v^2 = E[\bar{n}^2(i)]$. When the filter works in the steady state, namely, $i$ becomes a large number, we assume $E[\bar{I}^{2l}(i)] = \big(E[\bar{I}^2(i)]\big)^l = \sigma_I^{2l}$ and $E[\bar{n}^{2l}(i)] = \big(E[\bar{n}^2(i)]\big)^l = \sigma_v^{2l}$.

Since the high power terms $E[|e_a(i)|^4]$ and $E[|e_a(i)|^6]$ can be neglected, we obtain the excess MSE as follows,

$$\begin{aligned} \xi_{ex} &= E[|e_a(i)|^2] \\ &= \frac{E[\mu^2(\infty)]E[||\mathbf{Br}(i)||^2]K_2}{2E[\mu(\infty)](\sigma_I^2 + \sigma_v^2) - E[\mu^2(\infty)]E[||\mathbf{Br}(i)||^2]K_1}. \end{aligned} \tag{27}$$

We assume that the power of residual interference at the output of the optimum beamformer is significantly lower than the output noise power, namely,

$\sigma_I^2 \ll \sigma_v^2$. Thus, we can simplify the expression for the excess MSE as follows

$$\begin{aligned}\xi_{ex} &= E[|e_a(i)|^2] \\ &\approx \frac{E[\mu^2(\infty)]E[||\mathbf{Br}(i)||^2](\sigma_v^6+\sigma_v^4+4\sigma_v^2+2)}{2E[\mu(\infty)]\sigma_v^2 - E[\mu^2(\infty)]E[||\mathbf{Br}(i)||^2](3+3\sigma_v^4)}.\end{aligned} \tag{28}$$

In order to compute the final excess MSE, we also need to derive the steady-state first order and second order statistical expressions for the variable step-size values. By employing the methodology in [30], when $i$ becomes a large number, we obtain the following:

$$E[\mu(\infty)] = E[\gamma(i)]\mathcal{P} + \frac{(1-\mathcal{P})}{E[\gamma(i)]E[||\mathbf{Br}(i)||^2]}, \tag{29}$$

$$E[\mu^2(\infty)] = E[\gamma(i)]\mathcal{P} + \frac{(1-\mathcal{P})}{E[\gamma(i)]E[||\mathbf{Br}(i)||^4]}, \tag{30}$$

where $\mathcal{P}$ denotes the probability of update at the steady-state, which is given by

$$\begin{aligned}\mathcal{P} &= Pr\{E[|e(i)|^2] > E[|\gamma(i)|^2]\} \approx Pr\{|e(i)| > E[\gamma(i)]\} \\ &\approx 2Q\Big(\frac{E[\gamma(i)]}{\sigma_v}\Big),\end{aligned} \tag{31}$$

where $i$ is a very large number, $Pr\{.\}$ denotes the probability, and $Q(x)$ is the complementary Gaussian cumulative distribution function [50] which is given by $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$. The expression of $E[\gamma(i)]$ for the PDB scheme at the steady state can be derived based on (13) and is given by

$$E[\gamma(i)] = \sqrt{\lambda}\sigma_n||\tilde{\mathbf{w}}_{opt}||. \tag{32}$$

By following the same approach and using (16), we have the expression of $E[\gamma(i)]$ for the PIDB scheme at the steady state:

$$E[\gamma(i)] \approx \sqrt{\psi}\sqrt{E[\nu(i)]} + \sqrt{\lambda}\sigma_n||\tilde{\mathbf{w}}_{opt}||. \tag{33}$$

From (15), when $i$ becomes a large number, we have $E[\nu(i)] = E[|\mathbf{w}_{opt}^H\mathbf{Br}(i)|^2] = \mathbf{w}_{opt}^H\mathbf{B}\big(\sum_{k=1}^{q-1}\mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k) + \sigma_n^2\mathbf{I}\big)\mathbf{B}^H\mathbf{w}_{opt}$. In the simulations, we will show the effectiveness of our derivation and approximation.

## 6. Simulations

In this section, we evaluate the performance of the proposed set-membership adaptive blind beamforming algorithms and compare them with the existing adaptive blind beamforming algorithms including the conventional adaptive SG beamforming algorithms based on the CM and MV criteria with GSC structures. We carried out simulations to assess the convergence performance of signal-to-interference-plus-noise ratio (SINR) against the number of snapshots. In the simulations, we assume that there is one desired user in the system and the related DOA is known by the receiver. Simulations are performed with a ULA containing $m = 16$ sensor elements with half-wavelength inter-element spacing. The DOAs are randomly generated with uniform random variables between 0 and 180 degrees for each experiment. The results are averaged by 1000 runs. We consider the binary phase shift keying (BPSK) modulation and set $v = 1$.

In Fig. 3, we compare the proposed SM-CM-GSC adaptive beamforming algorithm with fixed bounds and that with time-varying bounds. We consider a scenario with $q = 6$ users with the same power level in the system. The input signal-to-noise ratio (SNR) is 15 dB. The initial values of the weight vector is given by $\mathbf{w}(0) = [1, 0, \ldots, 0]^T$. The coefficients for the PDB and PIDB schemes are given by $\rho = 0.98$, $\lambda = 2$, $\psi = 0.003$, $\gamma(0) = 0$ and $v(0) = 0$. For the fixed bound scheme, we set $\gamma = 0.1$, $\gamma = 0.6$ and $\gamma = 0.8$ to test the performance. The simulation results firstly illustrate that the SM-CM-GSC adaptive beamforming algorithm with $\gamma = 0.6$ provides a better performance compared to the other choices for the fixed bound. Secondly, we can see that the proposed beamforming algorithms with time-varying bound schemes outperform the beamforming algorithms with fixed bound schemes. Moreover, the performance with the PIDB scheme is slightly better than the performance with the PDB scheme. Due to the data-selective update feature the SM-CM-GSC adaptive algorithms with $\gamma = 0.1$, $\gamma = 0.6$ and $\gamma = 0.8$ can provide 72.9%, 27.6% and 14.8% update rates, respectively. The proposed beamforming algorithms with the PIDB and PDB schemes have 22.4% and 26.5% update rates, respectively.

Fig. 4 and Fig. 5 indicate the SINR convergence performance versus the number of snapshots for the proposed SM-CM-GSC adaptive algorithms and the conventional adaptive beamforming algorithms in the presence of different number of users. The input SNR is 15 dB. The number of users corresponding to the results in Fig. 4 and Fig. 5 are $q = 6$ and $q = 9$, respectively. The coefficients of the proposed adaptive beamforming algorithms with time-varying bound schemes are well tuned as the simulations of Fig. 3. The fixed bound of the SM-CM-GSC algorithm is $\gamma = 0.6$. The step-size values of the conventional SG adaptive CM-GSC and MV-GSC algorithms are tuned as $\mu = 0.005$. We note that all the parameters for the analyzed algorithms are optimized based on simulations. From the results, we can see that the proposed SM-CM-GSC adaptive beamforming algorithms with the PIDB and PDB schemes achieve the best convergence performance. While they only require around 20% of the time for filter parameter updates and can save significant computational resources. The performance of the minimum variance distortionless response (MVDR) beamforming solution is given as a reference.

Fig. 6 shows the convergence performance in a nonstationary scenario. The system starts with four users including one high-power level interferer with 3 dB above the desired user. At 1000 snapshots, three interferers including one user operating at 3 dB above the desired user's power level enter the system. From the results, we can see that the proposed SM-CM-GSC adaptive algorithm with the PIDB scheme achieves the best performance, followed by the SM-CM-GSC adaptive algorithm with the PDB scheme, the SM-CM-GSC adaptive algorithm with a fixed bound, the CM-GSC adaptive algorithm and the MV-GSC algorithm. The proposed algorithm is more robust to dynamic scenarios compared to the conventional SG-based algorithms. The SNR is 15dB. We set $\rho = 0.98$, $\lambda = 2$, $\psi = 0.003$, $\gamma(0) = 0$ and $v(0) = 0$. The fixed bound is chosen as $\gamma = 0.6$.

In the next simulation, we investigate the set-membership adaptive CM beamformers employing the DFP structure [40], and compare them with the proposed set-membership adaptive CM beamforming algorithms with the GSC structure. In particular, we investigate the SM-CM beamforming algorithms with the PIDB and PDB schemes for both DFP and GSC structures. The results as shown in Fig. 7 illustrate that the convergence performance of our proposed SM-CM-GSC algorithm with the PIDB scheme is slightly better than the performance of the SM-CM-DFP algorithm with the PIDB scheme. While the SM-CM-GSC algorithm with the PDB scheme outperforms the SM-CM-DFP algorithm with PDB scheme. In the experiment, the number of users is $q = 6$ and the input SNR is 15 dB. The coefficients of the GSC and DFP based beamforming algorithms were well tuned as the ones in the previous simulations.

We consider the convergence analysis of the proposed adaptive SM-CM-GSC beamformer with the PIDB scheme. The steady-state MSE between the desired and the estimated signal obtained through simulation is compared with the steady-state MSE computed via the expressions derived in Section 5. We verify that the analytical results (28), (29), (30) and (33) are able to predict the steady-state MSE. As the work proposed in [16], we use a scaled version of the Wiener filter to approximate the optimum CM-GSC solution. In this simulation of convergence analysis, we assume that three users having the same power level operate in the system. By comparing the curves in Fig. 8(a), it can be seen that as the number of snapshots increases and the simulated MSE converges to the analytical result, showing the usefulness of our analysis and assumptions. Fig. 8(b) shows the MSE performance versus the desired users SNR and a comparison between the steady-state analysis and simulation results. The simulation and analysis results agree well with each other.

In the final simulation results, we discuss the convergence analysis of the proposed SM-CM-GSC beamforming algorithm with the PDB scheme. Here, we verify that the analytical results (28), (29), (30) and (32) are able to provide an accurate prediction of the steady-state MSE. In this simulation, we assume that four users operate with the same power level in the system. Fig. 9(a) indicates that as the number of snapshots increases, the simulated MSE converges to the analytical result, showing the usefulness of our convergence analysis for the PDB scheme. Fig. 9(b) shows the effect that the desired users SNR has on the MSE. We also can see that the simulation and analysis results agree well with each other.

## 7. Conclusion

In this paper, we have proposed an adaptive blind set-membership beamforming algorithm with a GSC structure using the CM criterion. We have developed a SG-type algorithm based on the concept of SM filtering for adaptive implementation. We updated the filter weights only if the constraint cannot be satisfied. Moreover, two schemes of time-varying bounds have been proposed to blind beamforming with a GSC structure. We have also incorporated parameter and interference dependence to characterize the environment for improving the tracking performance of the proposed algorithm. For the proposed adaptive algorithm, we have investigated the convergence and derived expressions to predict the steady-state MSE. Simulation results have shown that the proposed blind SM beamforming algorithm with dynamic bounds achieves superior performance to previously reported methods at a reduced update rate.

## Appendix A. Analysis of the Optimization Problem

In this part, we discuss the convexity of the cost function which is expressed in (4). Without loss of generality, we assume that user 0 is the desired user. We rewrite the cost function as follows,

$$\begin{aligned} J_{CM} &= E\Big[\big(|\tilde{\mathbf{w}}^H(i)\mathbf{r}(i)|^2 - 1\big)^2\Big] \\ &= E\Big[\big(|y(i)|^2 - 1\big)^2\Big] \\ &= E\big[|y(i)|^4\big] - 2E\big[|y(i)|^2\big] + 1. \end{aligned} \tag{A.1}$$

Let us define $z_1(i) = \tilde{\mathbf{w}}^H(i)\mathbf{A}(\boldsymbol{\theta})\mathbf{b}(i) = \mathbf{s}^H\mathbf{b}(i)$ and $z_2(i) = \tilde{\mathbf{w}}^H(i)\mathbf{n}(i)$, where $\mathbf{s} = [s_0, \ldots, s_{q-1}]^T$ and $s_k = \mathbf{a}^H(\theta_k)\tilde{\mathbf{w}}(i)$, $k = 0, \ldots, q-1$. By letting $D = s_0 s_0^* = v^2$ and $\bar{\mathbf{s}} = [s_1, \ldots, s_{q-1}]^T$, we obtain

$$J_{CM} = J_1(\bar{\mathbf{s}}) + \sigma_n^2 J_2\big(\tilde{\mathbf{w}}\big), \tag{A.2}$$

where

$$J_1(\bar{\mathbf{s}}) = 2(D + \bar{\mathbf{s}}^H\bar{\mathbf{s}})^2 - \big(D^2 + \sum_{k=1}^{q-1} s_k^4\big) - 2(D + \bar{\mathbf{s}}^H\bar{\mathbf{s}}) + 1, \tag{A.3}$$

$$J_2\big(\tilde{\mathbf{w}}\big) = \big(4(D + \bar{\mathbf{s}}^H\bar{\mathbf{s}}) - 2 + 3\sigma_n^2\tilde{\mathbf{w}}^H\tilde{\mathbf{w}}\big)\tilde{\mathbf{w}}^H\tilde{\mathbf{w}}. \tag{A.4}$$

In order to evaluate the convexity of $J_{CM}$, we compute its Hessian matrix by using the rule $\mathbf{M} = \frac{\partial}{\partial\tilde{\mathbf{w}}^H}\frac{\partial J_{CM}}{\partial\tilde{\mathbf{w}}}$ which yields $\mathbf{M} = \mathbf{M}_1 + \sigma_n^2\mathbf{M}_2$, where

$$\mathbf{M}_1 = 4\bar{\mathbf{A}}\big[(D - 1/2)\mathbf{I} + \bar{\mathbf{s}}^H\bar{\mathbf{s}}\mathbf{I} + \bar{\mathbf{s}}\bar{\mathbf{s}}^H - \mathrm{diag}\big(|s_1|^2, \ldots, |s_{q-1}|^2\big)\big]\bar{\mathbf{A}}^T, \tag{A.5}$$

$$\begin{aligned} \mathbf{M}_2 = (4D - 2)\mathbf{I} &+ 6\sigma_n^2\big(\tilde{\mathbf{w}}^H\tilde{\mathbf{w}}\mathbf{I} + \tilde{\mathbf{w}}\tilde{\mathbf{w}}^H\big) + 4\big(\tilde{\mathbf{w}}^H\bar{\mathbf{A}}\bar{\mathbf{A}}^H\tilde{\mathbf{w}}\mathbf{I} + (\bar{\mathbf{A}}\bar{\mathbf{A}}^H)^T\tilde{\mathbf{w}}^H\tilde{\mathbf{w}} \\ &+ (\tilde{\mathbf{w}}\tilde{\mathbf{w}}^H\bar{\mathbf{A}}\bar{\mathbf{A}}^H)^T + (\tilde{\mathbf{w}}^H\bar{\mathbf{A}}\bar{\mathbf{A}}^H\tilde{\mathbf{w}})^T\big), \end{aligned} \tag{A.6}$$

where $\bar{\mathbf{A}} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_{q-1})]$.

The matrix $\mathbf{M}$ is positive definite if $\boldsymbol{\alpha}^H\mathbf{M}\boldsymbol{\alpha} > 0$ for any nonzero $(q-1)\times 1$ vector $\boldsymbol{\alpha}$. The second, third and fourth terms for $\mathbf{M}_1$ in (A.5) yield the positive definite matrix $4\big(\bar{\mathbf{s}}\bar{\mathbf{s}}^H + \mathrm{diag}\big(|s_1|^2,\ldots,|s_{q-1}|^2\big)\big)$, while the first term provides the condition $D = v^2 \geq 1/2$ that ensures the convexity of $J_1(\bar{\mathbf{s}})$. Therefore, when $\sigma_n^2 = 0$, the function $J_{CM}$ is convex. Since $J_{CM}$ is continuous in terms of $\sigma_n^2$, we may assume that the extrema of the cost function in noisy case can be deduced for small $\sigma_n^2$ by a slight perturbation of the noiseless extrema [17]. For the matrix $\mathbf{M}_2$ in (A.6), it is easily seen that we can select a sufficiently large value of $D$ such that $\mathbf{M}_2$ is positive definite in any bounded region. Recalling that $D = v^2$, we obtain that with properly selecting the constant $v$, $\mathbf{M}$ is positive definite in any bounded region, which results in the cost function $J_{CM}$ being strictly convex. The algorithm is then able to reach the global minima under these assumptions.

## Appendix B. Proof of (A.3) and (A.4)

We know that the cost function can be expressed as follows,

$$J_{CM} = E[|y(i)|^4] - 2E[|y(i)|^2] + 1, \tag{B.1}$$

where

$$y(i) = z_1(i) + z_2(i) = \mathbf{s}^H\mathbf{b}(i) + \tilde{\mathbf{w}}^H(i)\mathbf{n}(i). \tag{B.2}$$

In order to further investigate the cost function, we need to assess $E[|y(i)|^4]$ and $E[|y(i)|^2]$. By assuming that the source signals and the complex Gaussian noise are independent and identically distributed, we have

$$E[|y(i)|^4] = E[|z_1(i)|^4] + E[|z_2(i)|^4] + 4E[|z_1(i)|^2|z_2(i)|^2], \tag{B.3}$$

$$E[|y(i)|^2] = E[|z_1(i)|^2] + E[|z_2(i)|^2]. \tag{B.4}$$

Because the source signal takes on the value +1 with probability 0.5 or the value −1 with the same probability, we have

$$\begin{aligned} E[|z_1(i)|^4] = E[(\mathbf{s}^H\mathbf{b}\mathbf{b}^H\mathbf{s})^2] &= E\left[\left(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} s_i^* b_i b_j s_j\right)^2\right] \\ &= E\left[\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{l=0}^{q-1}\sum_{n=0}^{q-1} b_i b_j b_l b_n s_i^* s_j s_l^* s_n\right]. \end{aligned} \tag{B.5}$$

Since $s$ and $b$ are independent, we have $E[b_i] = 0$ for $\forall i$ and $E[b_i b_j] = 0$ for $i \neq j$, the only non-zero terms in the sum arise when the product of four values of $b$ at various times can be grouped into two pairs, i.e., $E[b_i b_j b_l b_n] = E[b_1^2 b_2^2]$ with $b_1 \neq b_2$ permitted. Exactly three grouping are possible from a set of four that gives a particular two pairs with different time arguments for the two pairs. However, only one possibility has them all the same. Thus,

$$E\big[|z_1(i)|^4\big] = \sum_{i=0}^{q-1}|s_i|^4 + 2\sum_{i=0}^{q-1}\sum_{l=0,\neq i}^{q-1}|s_i|^2|s_l|^2 + \sum_{i=0}^{q-1}\sum_{j=0,\neq i}^{q-1} s_i s_j^* s_i s_j^*, \tag{B.6}$$

because the $s$ are sequentially uncorrelated, $E[s_i s_j^* s_i s_j^*] = E[s_i s_i]E[s_j^* s_j^*] = 0$ for $i \neq j$. We use the observation to convert (B.6) to

$$E\big[|z_1(i)|^4\big] = \sum_{i=0}^{q-1}|s_i|^4 + 2\sum_{i=0}^{q-1}\sum_{l=0,\neq i}^{q-1}|s_i|^2|s_l|^2. \tag{B.7}$$

Thus, we can obtain

$$J_1(\bar{\mathbf{s}}) = 2(\mathbf{s}^H\mathbf{s})^2 - \sum_{k=0}^{q-1} s_k^4 - 2\mathbf{s}^H\mathbf{s} + 1, \tag{B.8}$$

$$J_2(\tilde{\mathbf{w}}) = (4\mathbf{s}^H\mathbf{s} - 2 + 3\sigma_n^2\tilde{\mathbf{w}}^H\tilde{\mathbf{w}})\tilde{\mathbf{w}}^H\tilde{\mathbf{w}}. \tag{B.9}$$

## Appendix C. Derivation for (12)

By imposing the condition to update whenever $e^2(i) > \gamma^2(i)$, we can obtain $\mu(i)$ in order to compute $\mathbf{w}(i+1)$ by projecting $\mathbf{w}(i)$ onto $\mathcal{H}_i$, i.e., the set of all $\mathbf{w}$ that satisfy:

$$\sqrt{1-\gamma(i)} \leq |(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w})^H\mathbf{r}(i)| \leq \sqrt{1+\gamma(i)}. \tag{C.1}$$

The set comprises two parallel hyper-strips in the parameter space. For case 1): $|y(i)| = |(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}(i))^H\mathbf{r}(i)| > \sqrt{1+\gamma(i)}$, $\mathbf{w}(i)$ is closer to the hyperplanes defined by $|(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w})^H\mathbf{r}(i)| = \sqrt{1+\gamma(i)}$ than to the ones defined by $|(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w})^H\mathbf{r}(i)| = \sqrt{1-\gamma(i)}$. By employing (11) we have

$$|y(i) + \mu(i)\big(\mathbf{Br}(i)y^*(i) - |y(i)|^2\mathbf{Br}(i)y^*(i)\big)^H\mathbf{Br}(i)| = \sqrt{1+\gamma(i)}, \tag{C.2}$$

which results in the following

$$\mu(i) = \Big(1 - \frac{\sqrt{1+\gamma(i)}}{|y(i)|}\Big)\frac{1}{(\mathbf{r}^H(i)\mathbf{B}^H|y(i)|^2 - \mathbf{r}^H(i)\mathbf{B}^H)\mathbf{Br}(i)}. \tag{C.3}$$

For case 2): $|y(i)| = |(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w}(i))^H\mathbf{r}(i)| < \sqrt{1-\gamma(i)}$, $\mathbf{w}(i)$ is closer to the hyperplanes defined by $|(v\mathbf{a}(\theta_0) - \mathbf{B}^H\mathbf{w})^H\mathbf{r}(i)| = \sqrt{1-\gamma(i)}$. By following the same approach we have

$$\mu(i) = \Big(1 - \frac{\sqrt{1-\gamma(i)}}{|y(i)|}\Big)\frac{1}{(\mathbf{r}^H(i)\mathbf{B}^H|y(i)|^2 - \mathbf{r}^H(i)\mathbf{B}^H)\mathbf{Br}(i)}. \tag{C.4}$$

Therefore, the expressions for $\mu(i)$ can be summarized in (12).

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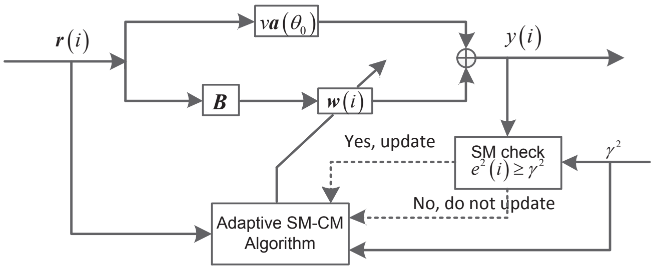


Figure 1: Proposed SM-CM-GSC adaptive beamformer structure.

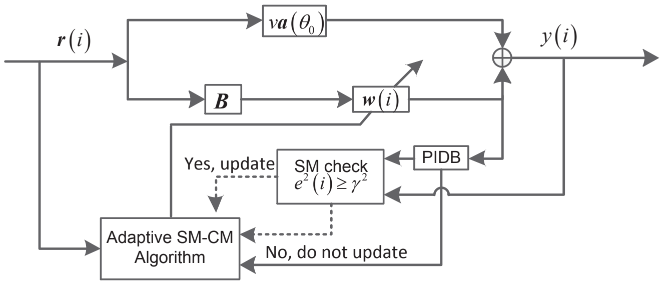


Figure 2: Proposed SM-CM-GSC adaptive beamformer structure with PIDB scheme.

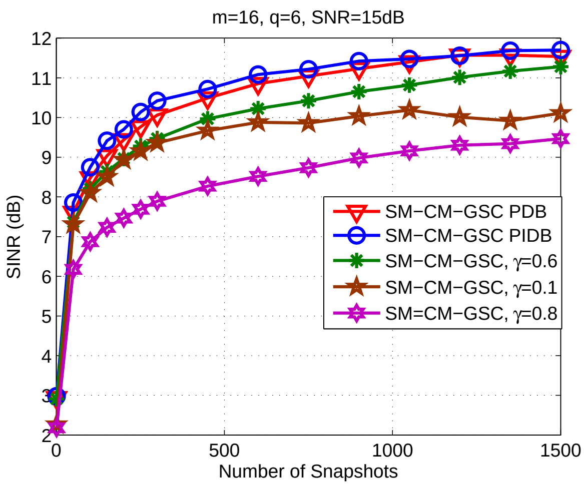


Figure 3: Output SINR versus the number of snapshots. The number of users is $q = 6$. The update rates for the proposed SM-CM-GSC algorithms with PIDB and PDB are 22.4% and 26.5%, respectively.

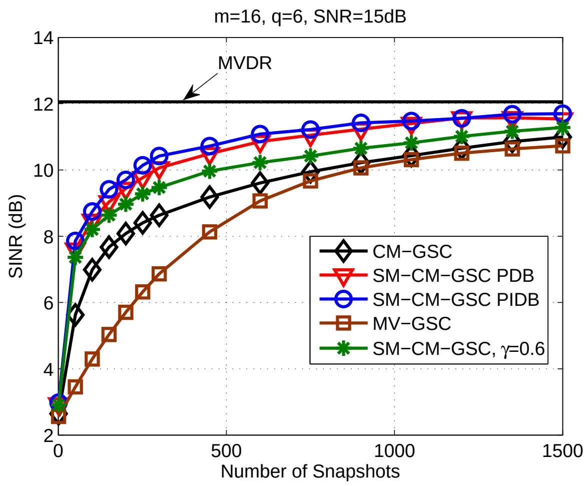


Figure 4: Output SINR versus the number of snapshots. The number of users is $q = 6$. The update rates for the proposed SM-CM-GSC algorithms with PIDB and PDB are 22.4% and 26.5%, respectively.

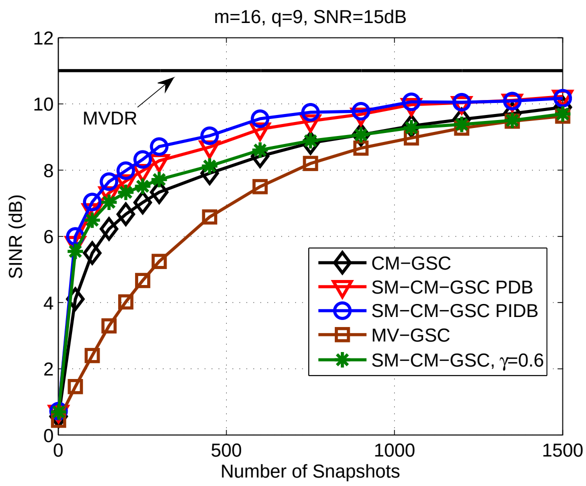


Figure 5: Output SINR versus the number of snapshots. The number of users is $q = 9$. The update rates for the proposed SM-CM-GSC algorithms with PIDB and PDB are 23% and 27%, respectively.

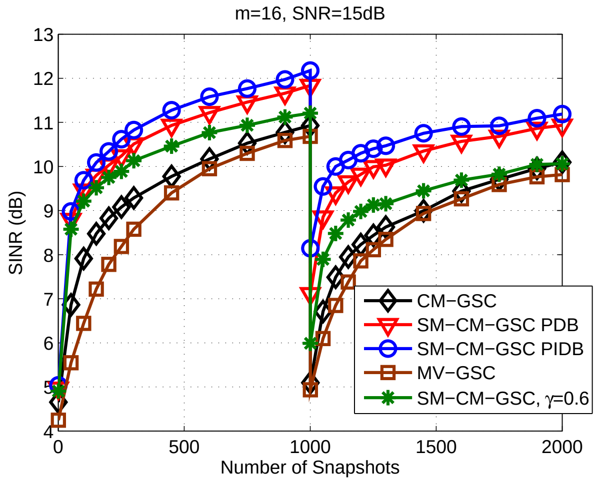


Figure 6: Output SINR versus the number of snapshots in a nonstationary scenario. SNR=15 dB. The update rates for the proposed SM-CM-GSC algorithms with PIDB and PDB are 25% and 30%, respectively.

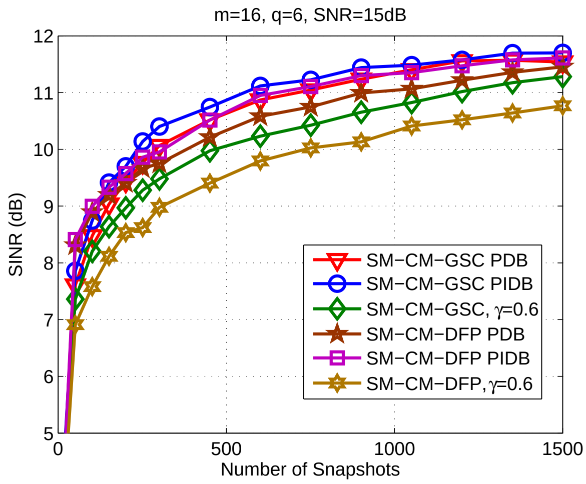


Figure 7: Output SINR versus the number of snapshots. SNR=15 dB. The number of users is $q = 6$. The update rates for the proposed adaptive SM-CM-GSC algorithms with PIDB and PDB are 22.4% and 26.5%. The update rates for the adaptive SM-CM-DFP algorithms with PIDB and PDB are 18.7% and 25.4%. The update rates for the adaptive SM-CM-GSC and SM-CM-DFP algorithms with a fixed error bound are 27.6% and 27%, respectively.

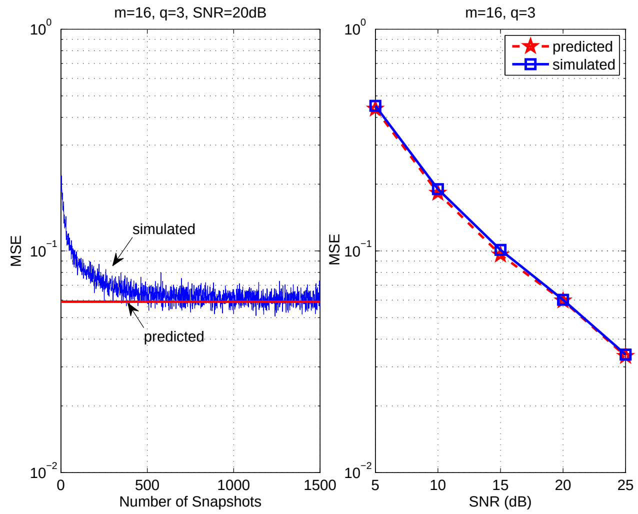


Figure 8: Analytical MSE versus simulated performance for convergence analysis of the proposed SM adaptive beamforming algorithm with PIDB scheme. (a) The number of users is $q = 3$, SNR=20 dB. (b) The number of users is $q = 3$.

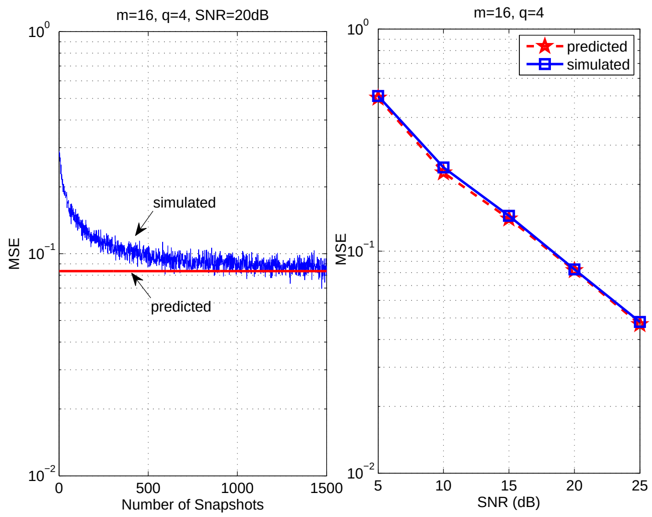


Figure 9: Analytical MSE versus simulated performance for convergence analysis of the proposed SM adaptive beamforming algorithm with PDB scheme. (a) The number of users is $q = 4$, SNR=20 dB. (b) The number of users is $q = 4$.