Information Theory and Channel Coding

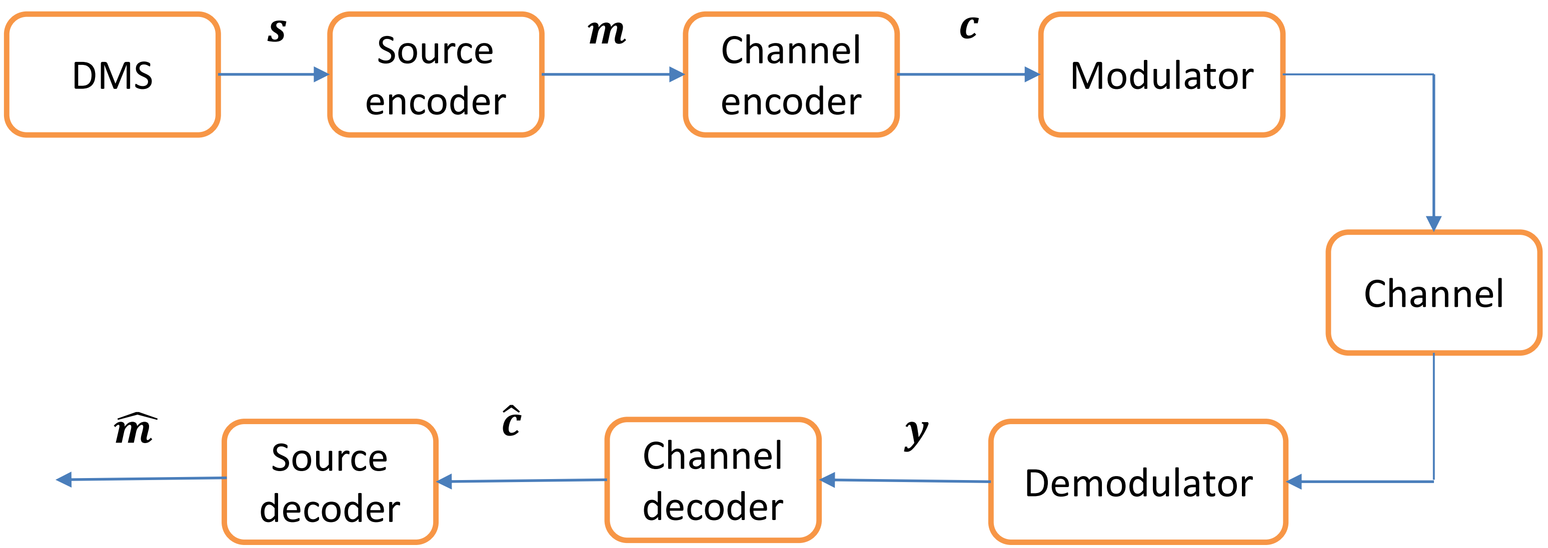
Prof. Rodrigo C. de Lamare
CETUC, PUC-Rio, Brazil
delamare@cetuc.puc-rio.br

IV. Channel coding

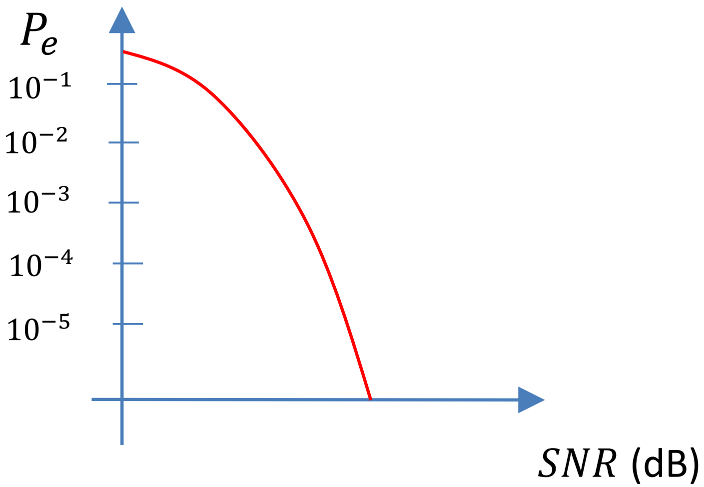
- In this chapter, we study the need for channel coding, derive the channel coding theorem and examine implications of the channel coding theorem.
- In particular, we examine the fundamental limit of how reliably information can be transmitted over a channel given some key parameters.
- We present a mathematical model of a digital communication system and how it can benefit from channel coding.
- We derive the channel coding theorem of Shannon using an approach based on the Markov inequality.
- We then examine implications of the channel coding theorem and how the probability of error of transmitted symbols can be made arbitrarily small.

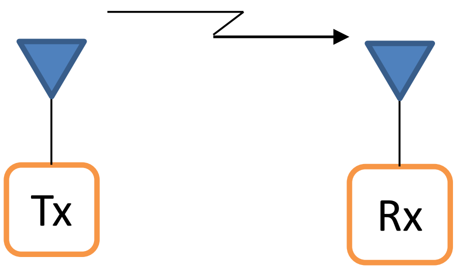
A. Digital communications model

- Digital transmission over a channel with capacity C involves several operations such as source coding, channel coding, modulation and decoding.



- Reliability is an important goal in digital communications that is often measured in terms of probability of symbol error P_e.

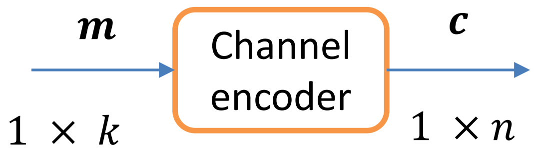




- In order to obtain reliable communication links and transmission, we need to employ channel coding.

- Channel coding increases the resistance against channel errors in digital transmissions.

- The basic idea of channel coding is to introduce redundancy.

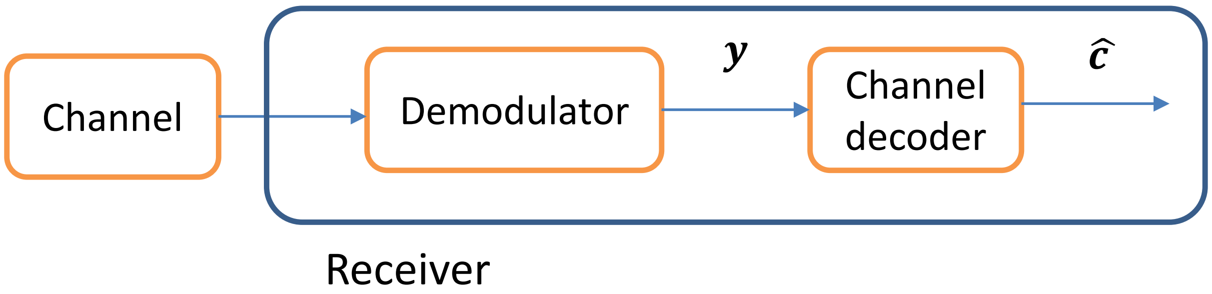


- A message $\boldsymbol{m}$ with k bits is mapped into a codeword $\boldsymbol{c}$ with n code bits, which is then transmitted.

- This redundancy translates into the code rate

$$R = \frac{k}{n}, \quad 0 < R < 1$$

- The receiver must deal with thermal noise often modelled as additive Gaussian noise and with the inverse mapping/decoding.



- Fundamental question:
    - Is there any channel coding scheme that allows transmission of messages with probability of error smaller than a small positive number ϵ ?

B. Channel coding theorem

For a discrete memoryless channel with capacity C that transmits information at a rate $R \leq C$ there exists a coding scheme in which the probability of error can be made arbitrarily small, that is,

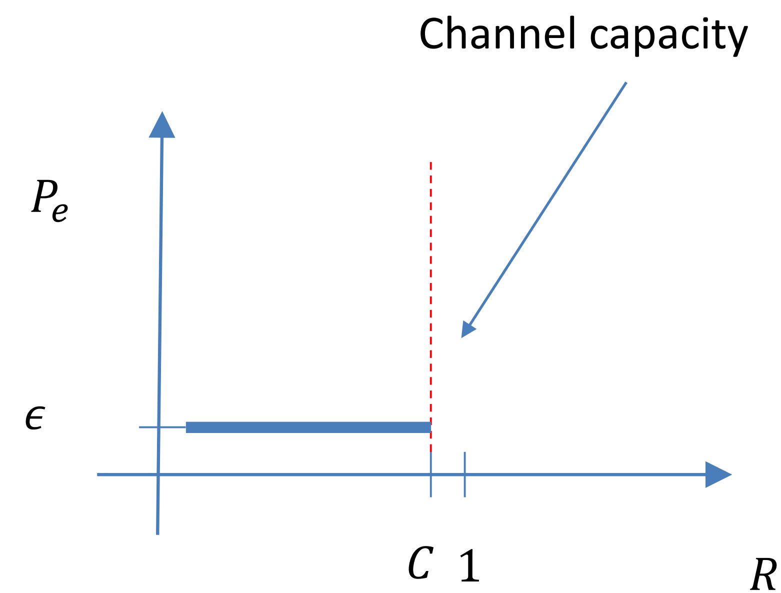
$$P_e \to \epsilon$$

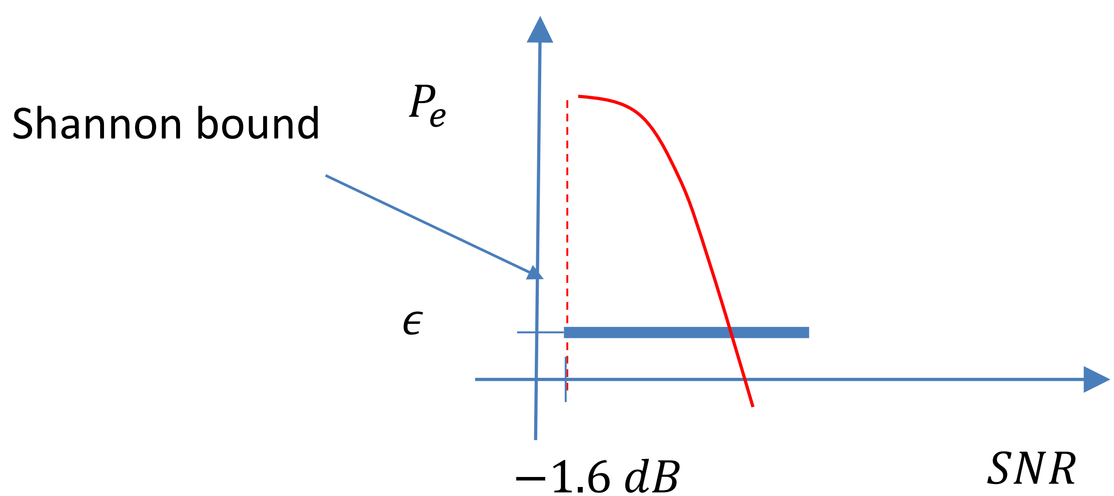
when the block length $n \to \infty$. This is known as achievability.

Conversely, if $R > C$ there is no coding scheme capable of delivering a P_e arbitrarily small. This is known as the converse theorem.

Interpretation of the theorem

- For code rate $R \leq C$ we can transmit information with arbitrarily low P_e.
- The theorem considers random codes but powerful channel codes could be designed close to capacity.
- Alternatively, a designer could use lower rates and approach the Shannon bound.





C. Proof of the channel coding theorem

- Standard proof in textbooks:
  - Based on joint typicality -> generation of long sequences with certain properties.
  - Use of the asymptotic equipartition property (AEP): analog of the law of large numbers.
  - According to AEP, typical sets of sequences of random variables are generated with equally probable elements.
- Our approach: based on Markov's inequality

Yuval Lomnitz, Meir Feder, "A simpler derivation of the coding theorem", https://arxiv.org/pdf/1205.1389.pdf

- Consider the Markov inequality of a random variable x given by

$$P(x \geq t) \leq \frac{E[x]}{t}$$

Let us also consider the following assumptions:

- Channel codes are assumed random: $\boldsymbol{x} = [x_1 \ldots x_n]$

- Entries of x are independent and identically distributed (i.i.d.) random variables, which yield the joint pdf

$$p_{\boldsymbol{x}}(\boldsymbol{X}) = \prod_{i=1}^{n} p_{x_i}(X_i)$$

- System model:

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n}$$
$$= [\mathrm{y}_1 \dots \mathrm{y}_\mathrm{n}]$$

- The noisy codeword $\boldsymbol{y}$ at the output of the channel is random and its elements are i.i.d., i.e.,

$$p_{\boldsymbol{y}}(\boldsymbol{Y}) = \prod_{i=1}^{n} p_{y_i}(Y_i)$$

- We assume that maximum likelihood decoding is employed:

$$\hat{\boldsymbol{x}} = \arg\max p_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{Y}|\boldsymbol{X})$$

Probability of error over a set of codes:

- Let $\boldsymbol{X}_m, m = 1,2,\dots,2^{nR}$ be the independent codes of $\boldsymbol{x}, \boldsymbol{y}$.

- Consider the event E_m where $\boldsymbol{X}_m$ leads to the inequality on the a posteriori probability

$$P_{y|x}(\boldsymbol{Y}|\boldsymbol{X}_m) \geq P_{y|x}(\boldsymbol{Y}|\boldsymbol{X})$$

- Therefore, we have

$$P(E_m|X,Y) = P\left(P_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{Y}|\boldsymbol{X}_m) \geq P_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{Y}\,|\boldsymbol{X})\middle|\boldsymbol{Y}\,,\boldsymbol{X}\right)$$

- Further developing the previous expression, we obtain

$$P(E_m|X,Y) = P\left(P_{y|x}(\boldsymbol{Y}|\boldsymbol{X}_m) \geq P_{y'|x}(\boldsymbol{Y}\,|\boldsymbol{X})\middle|\boldsymbol{Y}\,,\boldsymbol{X}\right)$$

$$\overset{\text{Markov}}{\leq} \frac{\mathrm{E}\left[P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})|\boldsymbol{Y}\,,\boldsymbol{X}\right]}{P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})}$$

$$= \sum_{\boldsymbol{X}_m \in \mathcal{X}^n} \frac{\mathrm{E}\left[P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})|\boldsymbol{Y}\,,\boldsymbol{X}\right]}{P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})}$$

$$= \sum_{\boldsymbol{X}_m \in \mathcal{X}^n} \frac{P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})P_{x}(\boldsymbol{X}_m)}{P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})}$$

$$= \frac{P_{y}(\boldsymbol{Y})}{P_{y\,|x}(\boldsymbol{Y}\,|\boldsymbol{X})}$$

- Using the union bound, the probability of error conditioned on $\boldsymbol{x}, \boldsymbol{y}$ is bounded by

$$
\begin{aligned}
P_{e|x,y} &\leq P\left\{\bigcup_{i=1}^{2^{nR}} E_m|\boldsymbol{X},\boldsymbol{Y}\right\} \\
&\leq 2^{nR} P(E_m|\boldsymbol{X},\boldsymbol{Y}) \\
&\leq 2^{nR} \frac{P_{\boldsymbol{y}}(\boldsymbol{Y})}{P_{\boldsymbol{y}\,|\boldsymbol{x}}(\boldsymbol{Y}\,|\boldsymbol{X})}
\end{aligned}
$$

- Then, we analyse the behaviour of $P_{e|x,y}$ for the DMC channel.

- Using the law of large numbers, we have

$$\frac{1}{n}\log_2 P(E_m|\boldsymbol{X},\boldsymbol{Y}) = \frac{1}{n}\log_2 \frac{P_{\boldsymbol{y}}(\boldsymbol{Y})}{P_{\boldsymbol{y}\,|\boldsymbol{x}}(\boldsymbol{Y}\,|\boldsymbol{X})}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\log_2 \frac{P_{y_i}(Y_i)}{P_{y_i\,|x_i}(Y_i\,|X_i)}$$

$$\xrightarrow[n\to\infty]{\text{in prob.}} E\left[\log_2 \frac{P_{y}(Y)}{P_{y\,|x}(Y\,|X)}\right] \triangleq -I(x,y), \text{ bits}$$

where x, y are two random variables that are distributed according to $p_{y\,|x}(Y\,|X)p_x(X)$.

- From the law of large numbers, it follows that for any $\epsilon, \delta > 0$ there is a sufficiently large n such that with probability $(1-\epsilon)$ we have

$$\frac{1}{n}\log_2 \frac{P_y(Y)}{P_{y\,|x}\,(Y\,|X)} \leq \; \delta - I(x,y)$$

- When the above expression is satisfied then we have

$$\begin{aligned} P_{e|x,y} &\leq 2^{nR} 2^{n\left(\delta - I(x,y)\right)} \\ &= 2^{-nR(I(x,y)-\delta-R)} \end{aligned}$$

- The expression $P_{e|x,y}$ can be averaged to obtain a bound on the probability of symbol error P_e

- The probability of symbol error P_e is limited by the union bound which is given by

$$P_e \leq \epsilon + 2^{-nR(I(x,y)-\delta-R)},$$

which can be made arbitrarily small if $R < I(x, y)$ or equivalently $R \leq C$ for small ϵ and δ

- For $n \rightarrow \infty$ with R fixed, we have

$$P_e \leq \epsilon$$

D. Implications of the channel coding theorem

- Let us consider a repetition code used for digital transmission over a BSC with crossover probability $p = 10^{-2}$.

- For such a BSC with probability $p = 10^{-2}$ the capacity is given by

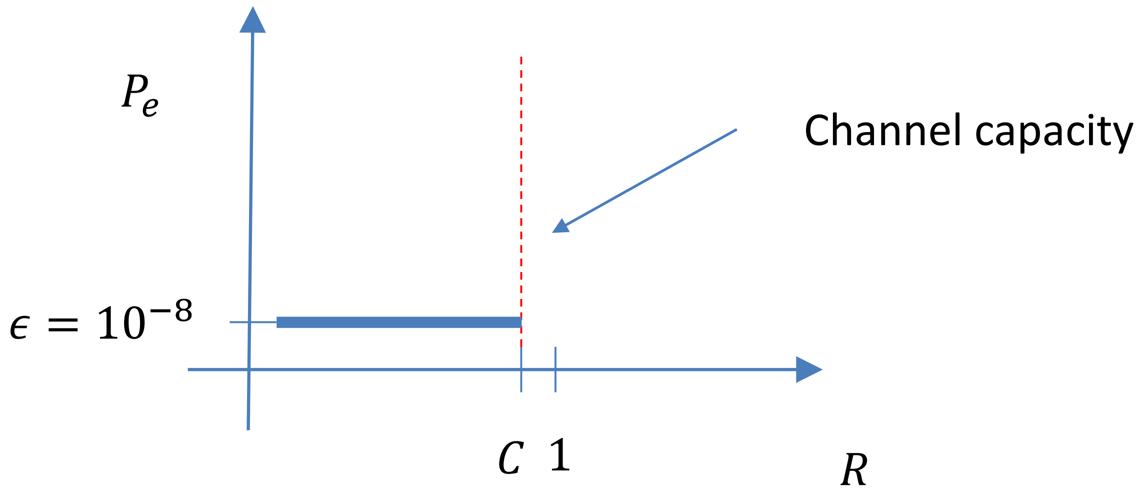
$$C = 1 - p\log_2 p - (1-p)\log_2(1-p)$$
$$= 0.9192$$

- Using the channel coding theorem, we know that for $\epsilon > 0$ and $R < 0.9192$ there exists a channel code with n sufficiently large, code rate R and a decoding algorithm that results in

$$P_e \leq \epsilon$$

Illustration

- For $\epsilon = 10^{-8}$, we have



- Consider a repetition code that works as follows:
  - Each bit of the message $\boldsymbol{m}$ is repeated multiple times.
  - For each bit (0 or 1) we repeat it n times, where $n = 2m + 1$ and n is an odd integer.
- The decoding of such code employs the majority logic decoding principle that works as follows:
  - If the number of 1s $\geq$ the number of 0s $\rightarrow$ the decoder decides for 1
  - If the number of 1s $<$ the number of 0s $\rightarrow$ the decoder decides for 0

- The probability of symbol error is given by

$$P_e = \sum_{i=m+1}^{n} \binom{n}{i} p^i (1-p)^{n-i},$$

where p is the crossover probability of the BSC channel.

- The probability of error is often used as a figure of merit and measured against the SNR or another useful quantity.

Performance

- The performance of the repetition code can be illustrated by measuring the probability of error P_e against the code rate R.

Code rate (R)	Proability of symbol error P_e
1	10^{-2}
$\frac{1}{3}$	3×10^{-4}
$\frac{1}{5}$	10^{-6}
$\frac{1}{7}$	4×10^{-7}
$\frac{1}{9}$	10^{-8}
$\frac{1}{11}$	5×10^{-10}