Enhanced EMD-Driven PEG Construction for Structured LDPC Codes

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***Abstract*—A novel algorithm to construct structured LDPC codes based on graph connectivity and cycle quality is proposed. The performance of the constructed QC-LDPC and IRA codes is investigated and a performance improvement in the error floor region is demonstrated. The code constructions provided are flexible in rate and length and demonstrate that the enhanced EMD-driven PEG-based approach proposed may be applied to a number of other structured LDPC code classes, given consideration of correct constraints of the construction.**

***Index Terms*—Channel coding, Low-density parity-check codes, Iterative decoding**

I. INTRODUCTION

Low-density parity-check (LDPC) codes, and Turbo Codes, have seen much interest due to their impressive performance combined with low complexity iterative decoding principles. Structured LDPC codes impose exploitable constraints on the parity-check matrix (PCM) of the LDPC code, allowing for a reduction in encoding complexity and further benefits in decoding. Quasi-cyclic (QC) LDPC codes are defined by a PCM made up of tiled circulant permutation matrices, allowing for complexity reduction and parallelisation [1]. Repeat accumulate (RA) codes are a class of codes which may be viewed as LDPC codes with a dual-diagonal substructure in their PCM, allowing for linear encoding complexity [2][3].

Asymptotic analysis of LDPC codes, through density evolution (DE) [4] provides a detailed view of the behaviour of LDPC codes under iterative decoding in the limit of very large block lengths. However, at short to medium lengths, this analysis fails to give a complete picture. Recent effort has focused on understanding the cycle structures present in the graph of these shorter length codes. To this end, the notions of stopping sets and trapping sets, also known as near-codewords, were developed. Stopping sets of a code graph entirely dictate the performance of the corresponding code on the binary erasure channel [5], while trapping sets account for further errors encountered on the additive white Gaussian noise (AWGN) channel [6].

In code construction, the girth has conventionally been viewed as the dominant feature of a particular graph influencing the performance of the corresponding code. Tian et al. [7] suggested, influenced by the idea of stopping sets, that the quality of particular cycles through their connection to the rest of the graph in certain cases has greater influence than cycle length. Based on the extrinsic message degree (EMD) of the cycle, the approximate cycle EMD (ACE) metric was then proposed as a measure of the graph connectivity of a cycle. The ACE metric was used with the powerful progressive edge growth (PEG) algorithm [8], a sub-optimal code construction which produces code graphs with large girth, in order to construct irregular LDPC codes with improved error floor performance [9]. A precise measure of cycle EMD was developed for use in a PEG-based construction method [10]. Other PEG-based designs have also been reported [11][12].

In this paper, we propose a PEG-based construction method which produces graphs with improved connectivity. The proposed construction avoids connections associated with large numbers of short cycles and uses the EMD information of all shortest-length cycles created by each potential edge placement. In this way, the connections made increase the likelihood of good overall graph connectivity and so decrease the likelihood of stopping set creation. This enhanced EMD-driven construction method shares the flexibility of other PEG-based designs [13][12]. This is demonstrated in the construction of irregular QC-LDPC and IRA codes which exhibit improved performance over previous constructions while maintaining the attractive properties of those structured classes of codes previously detailed.

The rest of the paper is organised as follows. In Section II the LDPC coding system is detailed and the challenges of short length code construction are discussed. In Section III the particular structures of QC-LDPC and IRA codes are briefly introduced. Section IV introduces the proposed enhanced EMD metric central to the structured code constructions proposed. Section V provides a detailed description of the proposed PEG-based construction using the criteria discussed in Section IV. Simulation results for the two structured code classes are presented in Section VI while Section VII concludes.

II. LDPC CODING SYSTEM AND PROBLEM STATEMENT

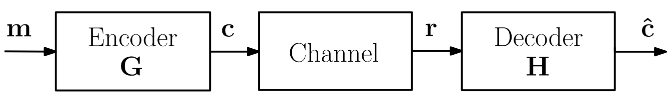

Fig. 1. Block diagram of the LDPC coding system

$$\mathbf{H}_{QC} = \left[\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

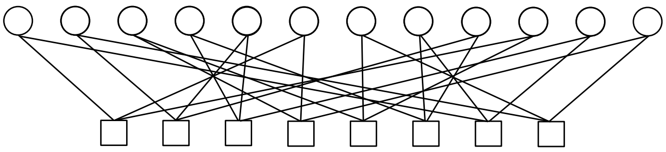

Fig. 2. Sample QC-LDPC PCM and associated graph

LDPC codes are iteratively decoded linear block codes. The PCM $\mathbf{H}$ fully characterises the code. The codeword $\mathbf{c}$ satisfies the constraint $\mathbf{c}\mathbf{H}^T = \mathbf{0}$. The LDPC coding system is illustrated in Fig. 1. For the AWGN channel considered, at the receiver we observe

$$r_\ell = (2c_\ell - 1) + n_\ell, \tag{1}$$

where r_ℓ is the ℓ*-th* entry of the received vector $\mathbf{r}$, $\ell = 1 : N$ and $n_\ell \sim \mathcal{N}(0, \sigma^2)$. The generator matrix $\mathbf{G}$ is derived from $\mathbf{H}$ and used in encoding. For QC-LDPC codes, $\mathbf{G}$ has QC structure and efficient encoding may be performed by shift registers. For IRA codes, encoding may be carried out by serial concatenated irregular repeater, interleaver, combiner and accumulator. The properties of these constituent operations are specified by the the PCM. The structures of both classes of codes are considered in more detail in Section III. Decoding is performed by the sum-product algorithm (SPA), operating on the graph of the code, with edges specified by the entries of $\mathbf{H}$. In Fig. 2, a sample is given of a particular PCM with the QC structure and its associated Tanner graph. An edge connects check node (CN) c_m to variable node (VN) v_ℓ in the Tanner graph of the code only if the the entry in $\mathbf{H}_{(m,\ell)} = 1$.

The code construction problem considered in this paper is to choose the explicit connections of the graph, satisfying all given code parameters, such as VN degree distribution, dimensions of the PCM and the required code structure with no repeated edge. The goal in constructing the code is then to ensure the best possible code performance.

For large block lengths, the performance of any particular member of an ensemble of LDPC codes tends towards the average performance of the ensemble [4]. This convenient fact allows for meaningful analysis of ensemble performance. It also means that in general for large block lengths, great care is not needed in choosing a particular graph from the ensemble for use in transmission.

It has been observed that asymptotic analysis does not completely describe the behaviour of the short length codes considered in this paper [7]-[10]. The threshold prediction of DE remains valid and the performance in the waterfall region of the bit error rate (BER) plot at low signal-to-noise ratio (SNR) behaves as expected. However, at short block lengths there is an observable error floor at higher SNRs, a decrease in improved performance with increasing SNR. This is well established, and has motivated the efforts mentioned in Section I [7]-[10]. The following gives a brief overview of the mechanisms through which the messages passed during SPA decoding become degraded, leading to poor performance.

The cycles present in short length graphs result in propagation of dependent messages in iterative decoding by SPA. The quality of messages passed is particularly harmed in the presence of short and interconnected cycles. In [7], an edge with only one connection to a given set of nodes, and therefore any cycles those nodes may participate in, is termed an extrinsic connection. The node it connects to is an extrinsic node. A set of VNs with no extrinsic connections is termed a stopping set and is known to be made up of one or more connected cycles. Enumeration of all stopping sets of a graph is a prohibitively complex problem, even for relatively short block lengths. As a result, attempts have been made to influence stopping set creation indirectly by ensuring short cycles had a high likelihood of numerous extrinsic connections, making the creation of a stopping set involving those cycles much less likely. This was initially done by means of the ACE metric [7]. This concept has been expanded upon to view the ACE properties of larger cycles [14], through the ACE spectrum of a code. In [14], it was clear that the number of short cycles a node is involved in, along with cycle length and its extrinsic connection properties, is important.

III. Structured LDPC Codes

A. QC-LDPC Codes

The PCM of QC-LDPC codes have the form

$$\mathbf{H}_{QC} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,t} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \cdots & \mathbf{A}_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{c,1} & \mathbf{A}_{c,2} & \cdots & \mathbf{A}_{c,t} \end{bmatrix}, \tag{2}$$

where each $\mathbf{A}_{a,b}$ is a $Q \times Q$ submatrix, either a circulant permutation or a null matrix. The positions of the nonempty submatrices and the shift from the zeroth position of the first entry in those nonempty matrices entirely determines the graph of the QC-LDPC code. It is clear from this where the advantages of these codes may be found in terms of complexity. In [13] the position of and the first entry in each $\mathbf{A}_{a,b}$ is determined by a modified PEG algorithm which operates on every *Q-th* column. The remaining entries in $\mathbf{A}_{a,b}$ are found by cyclic shifts. Care must be taken in the code construction of [13] that no more than one one entry is placed in one row or column of any one submatrix $\mathbf{A}_{a,b}$.

B. IRA Codes

IRA PCMs are structured as

$$\mathbf{H}_{IRA} = \left[\begin{array}{cc} \mathbf{H}_1 & \mathbf{H}_2 \end{array} \right], \tag{3}$$

where $\mathbf{H}_2$ is the dual diagonal matrix corresponding to the accumulator in the serial concatenated interpretation of IRA codes, while $\mathbf{H}_1$ is a low-density matrix with irregular column weights and row weights corresponding to the irregular repeat and combine stages of the serial encoder. As the form of $\mathbf{H}_2$ is predetermined, construction of an IRA code graph means setting the connections of $\mathbf{H}_1$ only. In a progressive columnwise construction, such as the one we deal with, we initialise the PCM as

$$\mathbf{H}_{init} = \left[\begin{array}{cc} \mathbf{H}_2 & \mathbf{0} \end{array} \right], \tag{4}$$

and start from the $(m+1)$-th column, where m is the number of CNs. Once construction is complete, the submatrices positions may be reversed to give the PCM the form of (3).

IRA codes offer linear complexity encoding and greater flexibility in degree distributions over QC-LDPC codes. However at low rates, the dual diagonal structure of the accumulator takes up a larger proportion of the PCM, causing greater deviation from DE derived degree distributions and harming performance. QC-LDPC codes offer flexibility in rate, provided the submatrix structure of (2) is adhered to. This structure also imposes some limitations on allowable degree distributions. The option of further parallelisation of both encoder and decoder makes QC-LDPC codes very useful for applications requiring low latency.

IV. Enhanced EMD Metric Calculation

The proposed enhanced EMD (EEMD) metric gives a measure of the likely effect of performance of a particular edge placement (c_i, v_j), allowing comparison of a set of CN candidates in a particular construction. CNs are compared first by the number of shortest paths they will create, with only those nodes creating the smallest number of shortest paths surviving to the next round of comparison. For each survivor, the average EMD across all shortest paths is computed, giving a measure of overall graph connectivity of the cycles associated with that prospective edge. The CN with the largest average EMD is selected. This process is described in more depth for the PEG-based construction in (5)-(7), Section V.

The novelty of the proposed construction is that it accounts for the case where multiple paths exist between root VN and a particular CN. This is a particularly important distinction at short block lengths, where multiple paths occur frequently. This issue was not considered in [10], where medium block lengths were considered and the distinction is less critical. Fig. 4 shows the performance of the QC-LDPC code constructed using the proposed PEG-based EEMD algorithm of Section V and a QC-LDPC PEG-based construction using the metric of [10]. The proposed construction outperforms both the previous EMD-based construction algorithm and the code constructed by the IPEG algorithm.

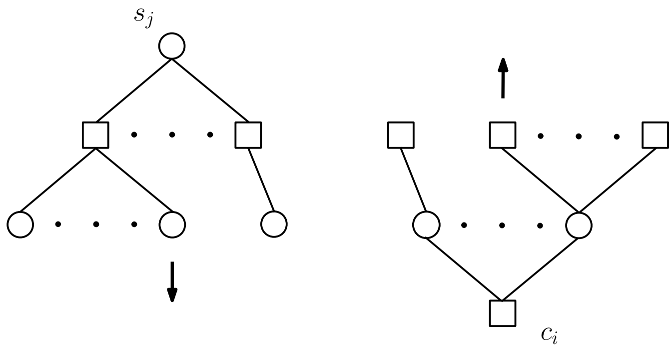

Fig. 3. PEG-like tree expansion

The computation of the proposed metric is carried out as follows. A PEG-like tree is expanded from both v_j and c_i, downwards and upwards, respectively, as illustrated in Fig. 3. It is clear that if a particular node is found at the same level of both downward and upward trees, then that node is present in a shortest path between v_j and c_i at that level. From knowledge of the VNs which appear in shortest paths at particular levels, it is then a trivial task to identify all distinct paths. The number of distinct paths each candidate CN is involved is the first piece of information we will use to separate good candidates from poor ones. Once all distinct shortest paths have been identified for a given candidate CN, the EMD of a path is simply the number of CNs connected to that path by a single link. For each extrinsic connection of a particular (CN,VN) path there will be a "1" in the sum of all columns, corresponding to VNs, which take part in the path. The EMD of that path is simply the number of "1"s present in the resulting vector. This is the second property of each candidate we will use in determining the best choice for placement.

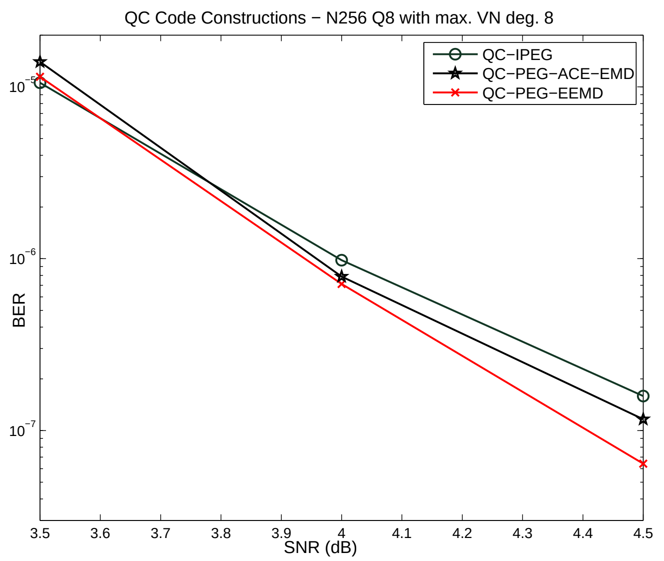

Fig. 4. Plot of performance of the proposed metric and previous work for PEG-based constructions

V. Proposed EMD-Driven PEG Design

The previously discussed metric calculation is versatile, it can be computed for any (CN,VN) pair. An exhaustive search of every CN for connection to a given VN, and repeated at every VN for the required number of connections to that VN would be prohibitively complex. It is necessary to produce a shorter list of candidate CNs. As previously discussed, the PEG algorithm is a sub-optimal graph construction algorithm which operates edge by edge and at each placement produces the longest cycle possible given the current graph configuration. This algorithm produces a list of CNs at maximum distance from the VN of interest and equal minimum current weight. This provides a good subset of CNs which can then be compared by means of the metric described in Section IV.

The set of CNs, provided by the PEG algorithm, at equal maximum distance from the root VN v_j is denoted $\overline{N_{v_j}^l}$ in the notation introduced in [8]. In the original PEG algorithm the successful candidate is taken at random among the minimum weight CNs of this set. This set of minimum weight furthest CNs is denoted C in the pseudocode below. In defining this set, the notation of [8] for the neighbourhood of a node has been used. Specifically, $N_{c_i}^1$ denotes the set of VNs reached by a tree emanating from the CN c_i and expanded only one level, meaning in this case the weight of the CN c_i. If there is more than one CN in the set C, then for each c_i in the set, all distinct paths from v_j to c_i are found as described in Section IV and the choice is made based on the number of shortest cycles which would be created by each candidate connection, and the average EMD of those shortest cycles which the candidate CN would participate in, if placement was made.

The set of candidates presented by the PEG algorithm is

$$C = \{c_i : |N_{c_i}^1| = \min_{c_x \in \overline{N_{v_j}^l}} |N_{c_x}^1|\}. \tag{5}$$

From this set the surviving CNs are

$$D = \{c_k : c_k \in C, P_k = \min_{x=1:|C|} P_x\}, \tag{6}$$

where $P_x, x = 1 : |C|$ is defined in Step 8 of the pseudocode of Algorithm 1. The CN finally chosen for placement, provided multiple options remain, is then

$$c_i : c_i \in D, \mathrm{Metric_i} = \max_{\mathrm{x=1:|D|}} \mathrm{Metric_x}, \tag{7}$$

where $\mathrm{Metric_x}$ for $x = 1 : |D|$, defined by Steps 14-18 of Algorithm 1, is the average of the EMD values of all shortest paths from the current VN of interest v_j to the x-*th* CN in the set $\mathbf{D}$. This gives a clearer indication of likely stopping set creation when there are many paths from v_j to c_i, as is often the case for short codes.

Algorithm 1 Proposed EMD-Driven PEG Design

1. Given the set $\overline{N_{v_j}^l}$:
2. The set $C = \{c_i : |N_{c_i}^1| = \min_{c_x \in \overline{N_{v_j}^l}} |N_{c_x}^1|\}$
3. **if** $|C| == 1$ **then**
4.   The edge (C, v_j) is placed
5. **else**
6.   **for** $y = 1 : |C|$ **do**
7.   The downward/upward PEG-like trees of Fig. 3 are expanded from v_j, C_y respectively.
8.   From this operation, all distinct paths are discovered for this VN, CN pair. Denote the number of paths P_y. Then the VNs making up the shortest paths are stored as the rows of the $P_y \times l$ matrix $\mathbf{V}_y$.
9.   **end for**
10.   The set $D = \{c_k : c_k \in C, P_k = \min_{x=1:|C|} P_x\}$
11.   **if** $|D| == 1$ **then**
12.   The edge (D, v_j) is placed
13.   **else**
14.   **for** $w = 1 : |D|$ **do**
15.   **for** $z = 1 : P_w$ **do**
16. $\mathrm{EMD_z} = \mathrm{sum}(\mathrm{sum}(\mathbf{H}(:, \mathbf{V}_\mathrm{w}(\mathrm{z}, :)), 2) == 1)$
17.   **end for**
18. $\mathrm{Metric_w} = \mathrm{mean}(\mathrm{EMD})$
19.   **end for**
20.   The chosen CN is then $c_i : c_i \in D$,
21. $\mathrm{Metric_i} = \max_{\mathrm{x=1:|D|}} \mathrm{Metric_x}$.
22.   The edge (c_i, v_j) is placed.
23.   **end if**
24. **end if**

VI. Simulation Results

Results are presented for irregular structured short-length LDPC codes for the rate $\frac{1}{2}$ case. The codes were constructed based on the DE derived VN degree distribution with maximum degree 8, presented in [4], Table II:

$$\lambda(x) = .30013x + .28395x^2 + .41592x^7 \tag{8}$$

As we consider PEG-based constructions only, the DE derived $\rho(x)$ is not considered, the graphs produced have near-regular check degree of the form:

$$\rho(x) = ax^b + (1 - a)x^{b-1}. \tag{9}$$

A number of necessary alterations were made to the VN degree distribution of (8), to account for the short lengths of the codes and the structured codes considered. Following [3], the number of weight-2 nodes of the distribution was limited to less than the number of CNs. For the QC-LDPC codes the distribution was further altered to take into account the submatrix structure of the PCM. For the rate $\frac{1}{2}$ IRA codes, $\mathbf{H}_1$ is constructed such that its columns have weights equal to the M higher degree VNs prescribed by (8). The other M VNs required by our constrained version of (8) correspond to the columns of the accumulator $\mathbf{H}_2$, with a single weight-1 entry required in order to allow efficient encoding.

For the BER plots presented, transmission was simulated on the AWGN channel. The decoder was operated to a maximum of 40 iterations and 100 block errors were gathered for each point in the plots. Improved performance is seen in the error floor region for both the QC-LDPC and IRA codes constructed by the proposed QC-PEG-EMD algorithm compared with both the IPEG-based constructions using the ACE metric and the original PEG-based constructions.

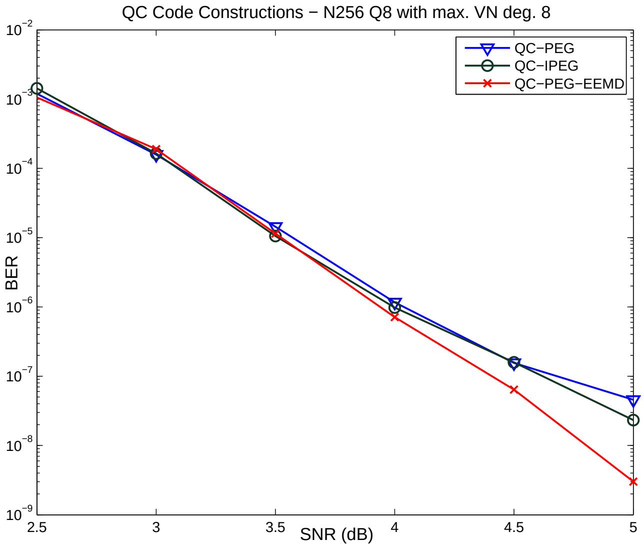

Fig. 5. Performance of QC-LDPC codes of different constructions

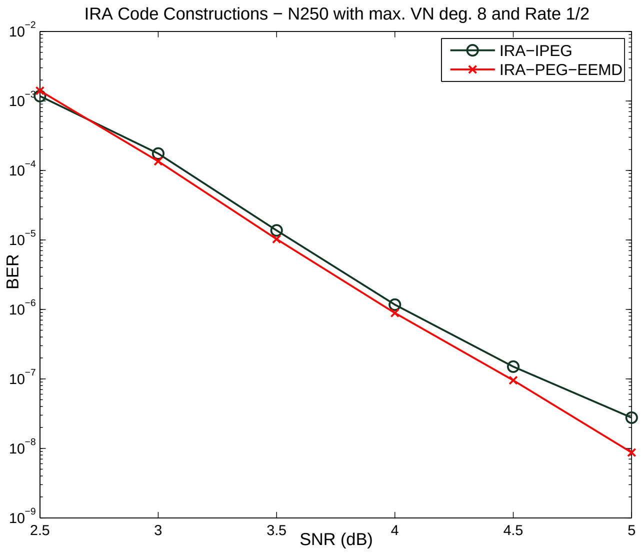

Fig. 6. Performance of IRA codes of different constructions

VII. Conclusion

For both types of structured LDPC codes an improved error floor performance is observed. This improvement is due to improved connections among potentially harmful short cycles, resulting from two decision criteria developed in order to compare placement choices, namely the number of shortest cycles which would be created by a given placement and the average EMD across those shortest cycles. This approach leads to a lower probability of stopping set creation.

The structured codes considered are very practical, particularly at the short lengths for which results are presented. The proposed EMD-driven design requires increased complexity compared to the PEG-based designs previously used to construct codes of these classes, however this extra effort is required only during the construction phase, during transmission the codes constructed by this method incur no extra cost and provide an improved performance at medium to high SNRs.

The versatility of the proposed design method should also be noted. It is flexible in rate, length and with appropriate constraints, structure. This has been observed for the Root-LDPC class of structured LDPC codes designed to achieve the diversity of the block fading channel [15]. Previous work has shown the success of PEG-designed Root-LDPC codes [12] and preliminary results indicate coding gains when the proposed method is used for code construction. Further investigation of the achievable gains of the proposed EMD-driven construction for Root-LDPC codes remains as a future work.

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